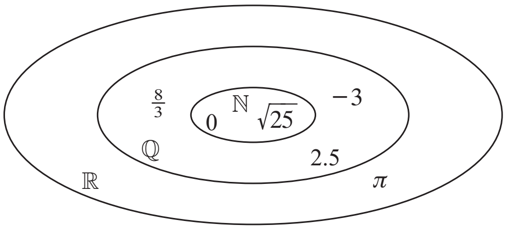
Short questions

1 **a** **i** $A = \{-1, 0\ 1, 2\}$ **ii** $B = \{2, 3, 5, 7, 11, 13\}$
iii $C = \{-\sqrt{8}, +\sqrt{8}\}$

b **i** false **ii** true **iii** true

2



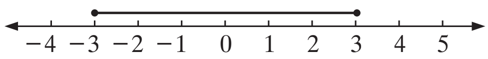
3 $A = \{0, 3, 6, 9\}$ $B = \{1, 2, 5, 10\}$

a $A \cap B = \{\ \}$

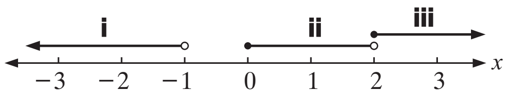
b $A \cup B = \{0, 1, 2, 3, 5, 6, 9, 10\}$

c $(A \cup B)' = \{4, 7, 8, 11\}$

4 **a**



b



5 $P = \{1, 3, 5, 7, 9, 11\}$ $Q = \{1, 2, 3, 4, 6, 12\}$
$R = \{5, 10\}$

a $P \cap Q = \{1, 3\}$ **b** $(P \cap Q) \cup R = \{1, 3, 5, 10\}$

c $Q \cap R = \{\ \}$ **d** $P' \cap (Q \cup R) = \{2, 4, 6, 10, 12\}$

6 **a** $n(P) = 22$ **b** $n(P \cup Q) = 47$

c $n(Q \cap R) = 13$ **d** $n((P \cup Q \cup R)') = 3$

e $n(Q') = 30$ **f** $n(P \cap R) = 0$

7 **a** If E is the set of students studying English and S those studying Spanish then

$$\begin{aligned} n(E \cup S) &= n(E) + n(S) - n(E \cap S) \\ &= 18 + 25 - 6 \\ &= 37 \end{aligned}$$

So, 37 IB students study English or Spanish.

b Let F be the set of students studying in French and E be those studying in English. If all students must study either in French or English then 100% study in one of these languages.

$n(F \cup E) = n(F) + n(E) - n(E \cap F)$

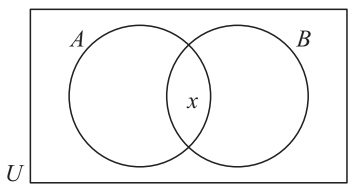
Using percentages, $100\% = 60\% + 76\% - x\%$ where $x\%$ is the percentage of students studying in both languages.

$x\% = (60 + 76 - 100)\% = 36\%$

i.e., 36% of students study in both languages.

c 20% of $25 = 5$

Let A be the set of students studying art and D be those studying drama. This information is summarised in the Venn diagram (x is the number studying both art and drama).



$n(U) = 25$

$$n(A \cup D) = 25 - 5 = 20$$

Now $n(A \cup D) = n(A) + n(D) - n(A \cap D)$

$\therefore\ 20 = 13 + 9 - x$

$\therefore\ x = 13 + 9 - 20 = 2$

So, 2 students study both art and drama.

8 $\sqrt{\dfrac{32.76}{3.95 \times 2.63}} = 1.775\,806\,022\ldots.$

a $= 1.776$ {correct to 3 d.p.}

b $= 2$ {nearest integer}

c $= 1.78$ {correct to 3 s.f.}

d $= 1.78 \times 10^0$ {standard form}

9 Speed of light $= 186\,280 \times 1.609$ km/sec
$= (186\,280 \times 1.609) \times 60$ km/min
$= 17\,983\,471.2$ km/min

a correct to 3 s.f. $= 18\,000\,000$ km/min
(**Note:** The first zero is significant.)

b In scientific notation, 1.80×10^7 km/min.

c Time for light to reach Earth is
$\dfrac{195\,000\,000}{18\,000\,000} = 10.8$ minutes.

10 **a** **i** Actual error is $5.65 - 5.645 = 0.005$ cm

ii Relative error is $\dfrac{0.005}{5.645} = 0.000\,886$

iii Relative percentage error is $0.000\,886 \times 100$
$= 0.0886\%$

b **i** Actual error is $70 \times 3.2\% = 2.24$ kph.

ii (70 ± 2.24)
minimum 67.76 kph, maximum 72.24 kph

11 **a** Error is $4 - 3.94 = 0.06$ m.

b Length of joined pipes $= 5 \times 3.94 = 19.7$ m.

c Error $= (5 \times 4) - (5 \times 3.94)$
$= 0.3$ m

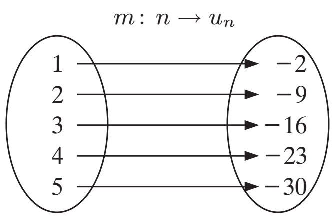
d Percentage error is $\dfrac{0.3}{19.7} \times 100 = 1.52\%$

12 **a** $-9 - (-2) = -7$
$-16 - (-9) = -7$

So, this is an arithmetic sequence with common difference $= -7$.

The sequence is $-2, -9, -16, -23, -30, \ldots\ldots$

b



c $$\begin{aligned} u_n &= u_1 + (n-1)d \\ &= -2 + (n-1)(-7) \\ &= 5 - 7n \end{aligned}$$

13 **a** $u_1 = 1(1+1) = 2,\quad u_2 = 2(2+1) = 6,$
$u_3 = 3(3+1) = 12$

b $u_{15} = 15(15+1) = 240$

c $n(n+1) = 600$ $\quad n^2 + n - 600 = 0$
$(n - 24)(n + 25) = 0$
$n = 24$ or -25

But $n > 0$, so the 24th term is 600.

14 a Arithmetic, hence

$$\begin{aligned}(k-166)-(-347) &= -185-(k-166)\\ k-166+347 &= -185-k+166\\ 2k &= -200\\ k &= -100\end{aligned}$$

b Common difference is $(-100-166)+347=81$

$$\begin{aligned}u_n &= -347+(n-1)81\\ &= -428+81n\end{aligned}$$

c $-428+81n=0$ if $n=\dfrac{428}{81}=5.28$

The 6th term has the first positive value (58).

15 a

$$\begin{aligned}u_1+5d &= 49 \quad \text{...... (1)}\\ u_1+14d &= 130 \quad \text{...... (2)}\\ \therefore\ 9d &= 81 \quad \{\text{subtracting (1) from (2)}\}\end{aligned}$$

Common difference $d=9$.

b $u_1+5\times 9=49$ First term is $u_1=4$.

c

$$\begin{aligned}4+(n-1)9 &= 300\\ 4+9n-9 &= 300\\ 9n &= 305\\ n &= 33.9....\end{aligned}$$

The first 33 terms are less than 300.

16 a

$$\begin{aligned}\tfrac{7}{2}(2u_1+(7-1)14) &= 329\\ 2u_1+84 &= 94\\ 2u_1 &= 10 \qquad \text{First term is 5.}\end{aligned}$$

b $\dfrac{n}{2}(2\times 5+(n-1)\times 14)=69\,800, \quad \therefore\ n=100$

17 a Common ratio $=\dfrac{6.75}{2.25}=3$

b $u_n=0.75\times 3^{n-1}$

c $S_n=\dfrac{0.75\times(3^{10}-1)}{3-1}\approx 22\,143$

18 a Common ratio $=\dfrac{4.5}{5}=0.9$

b Height of third bounce is $4.05\times 0.9=3.645$

c Distance travelled is $5+2(4.5)+2(4.05)+2(3.645)$
$=29.39$ m

19 a Half of $128=64$ clubs remain in the second round.
Half of $64=32$ clubs remain in the third round.

b $u_n=u_1r^{n-1}$

i.e., $u_n=128\times(0.5)^{n-1}$ (or 2^{8-n})
remaining in the nth round.

c There are 7 rounds needed to find a winner.

20 a $u_1r^{2-1}=14.5, \quad u_1r^{5-1}=1.8125$

Substitute for $u_1=\dfrac{14.5}{r}$

$$\begin{aligned}\therefore\ \frac{14.5\times r^4}{r} &= 1.8125\\ \therefore\ r^3 &= 0.125\\ \therefore\ r &= 0.5\end{aligned}$$

The common ratio is 0.5.

b Since $u_1(0.5)=14.5$
then $u_1=29$ and the first term is 29.

c $S_5=\dfrac{29(0.5^5-1)}{0.5-1}=56.1875$

21 a Population in 2005 is $1200\times 1.09^5=1846$.

b

$$\begin{aligned}1200\times 1.09^n &= 2500\\ \therefore\ 1.09^n &= \frac{2500}{1200}=\tfrac{25}{12}\\ \therefore\ n &\approx 8.52 \qquad \{\text{gdc}\}\end{aligned}$$

Population reaches 4500 during the year 2009.

c If $1200\times r^{10}=3200$

$$\begin{aligned}\text{then}\ r^{10} &= \frac{3200}{1200}=\tfrac{32}{12}\\ \therefore\ r &= (\tfrac{32}{12})^{\frac{1}{10}}\approx 1.103\end{aligned}$$

Rate of increase is 10.3%.

22 a

$$\begin{aligned}178-4n &= 7n+57\\ \Leftrightarrow\ 178-57 &= 7n+4n\\ \Leftrightarrow\ 121 &= 11n\end{aligned}$$

i.e., the sequences have a term in common if $n=11$.

The term they have in common is
$178-4\times 11=7\times 11+57=134$.

b Break even when $R=C$

$$\begin{aligned}\text{i.e.,}\ 25n &= 21\,000+7.5n\\ \Leftrightarrow\ 17.5n &= 21\,000\\ \text{i.e., if}\ n &= 1200\end{aligned}$$

Break even occurs when 1200 goods are produced and sold.

c

$$\frac{n(n+1)}{2}>435$$

$$\begin{aligned}\text{Solve}\ n(n+1) &= 870\\ n^2+n-870 &= 0\\ (n+30)(n-29) &= 0\\ n &= -30 \ \text{or}\ 29\end{aligned}$$

but $n>0$, so $n=29$

The sum exceeds 435 for $n>29$.

23 a **i** $s=16, \quad r=5$ **ii** $u_1=27, \quad d=5$

b $a+12p=20$ and $a+22p=34$

Using a gdc, $a=3.20$ and $p=1.40$.

24 a Let l be the length, then $2x+2l=80$
i.e., $2l=80-2x$
so the length is $40-x$

b

$$\begin{aligned}\text{Area rectangle} &= \text{length}\times\text{width}\\ &= (40-x)x\\ \text{i.e.,}\ A &= 40x-x^2\end{aligned}$$

c

$$\begin{aligned}40x-x^2 &= 375\\ \text{i.e.,}\ x^2-40x+375 &= 0\\ (x-15)(x-25) &= 0\\ x &= 15\ \text{or}\ 25\end{aligned}$$

The length of the rectangle is 15 or 25 cm.

25 a

$$\begin{aligned}70t-5t^2 &= 0\\ \text{i.e.,}\ 5t(14-t) &= 0\\ t &= 0\ \text{or}\ 14\end{aligned}$$

The rocket is in the air for 14 seconds.

b

$$\begin{aligned}70t-5t^2 &= 30\\ \text{i.e.,}\ -5t^2+70t-30 &= 0\end{aligned}$$

Using a gdc, $t=0.443$ or 13.6

The rocket was above 30 m between 0.443 and 13.6 sec.

Long questions

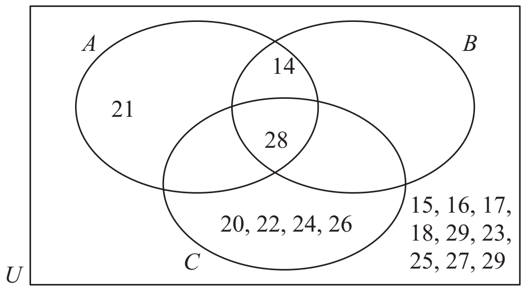
1 a i is false since '2' is not included in $\{x \in \mathbb{R} \mid x < 2\}$

ii is true

iii is false since $2^2 + 2 = 4$ is false

b i $A = \{14, 21, 28\}$, $B = \{14, 28\}$, $C = \{20, 22, 24, 26, 28\}$

ii



iii a $A \cap B \cap C = \{28\}$

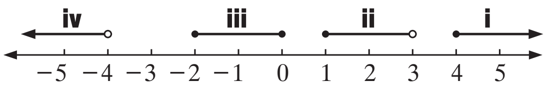
b $(A \cap B) \cup C = \{14, 20, 22, 24, 26, 28\}$

c $(A \cap B)' \cap C = \{20, 22, 24, 26\}$

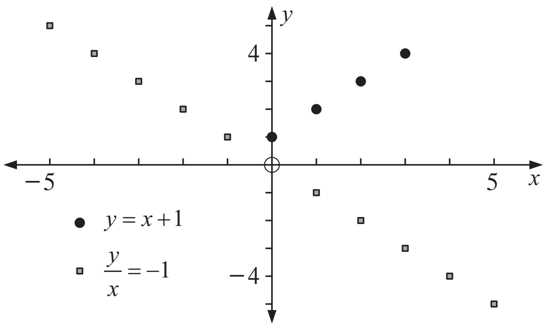
d $(A \cup C)' \cap B = \{\ \ \}$

c ii is false. Consider $p = 1$, $q = 2$. $1 - 2 \neq 2 - 1$

2 a



b



c Let $x = -2$, $y = -3$, then $x^2 = 4$ and $y^2 = 9$ and so $x > y$ but $x^2 < y^2$.

3 a $25.32 \times \dfrac{6.057}{2.4 \times \sqrt{5.14}} = 28.185\,67\ldots$

i $= 28.186$ {to 5 s.f.}

ii $= 28.2$ {to nearest tenth}

iii $= 30$ {to 1 s.f.}

b i Actual length covered is $3 \times 3.63 = 10.89$ m.

ii Error between actual length and stated length is

$10.89 - 10.80 = 0.09$ m

$$\text{Percentage error} = \frac{10.89 - 10.8}{10.89} \times 100 = 0.826\%$$

c i Minimum volume needed is

$10.89 \times 10.89 \times 0.095 = 11.27\ \text{m}^3$ (11.1)

Maximum volume needed is

$10.89 \times 10.89 \times 0.105 = 12.45\ \text{m}^3$ (12.5)

ii Planned volume $= 10.8 \times 10.8 \times 0.1$

$= 11.66\ \text{m}^3$ (11.7)

Difference in cost is $(12.45 - 11.66) \times 47.50$

$= 37.53$ Euro

iii Percentage difference in cost is

$$\frac{37.53}{11.66 \times 47.50} = 6.78\%$$

4 a i 120, 123, 126 **ii** 4, 11, 18

b i $120 + 3(n-1) = 4 + 7(n-1)$

$\therefore\ 120 + 3n - 3 = 4 + 7n - 7$

$\therefore\ 120 = 4n$ and so $n = 30$

The 30th term is the same.

ii $u_{30} = 120 + 3(30-1) = 4 + 7(30-1)$

$= 207$

c $120 + (n-1) \times 3 = 151$, $\therefore\ n = 11.3$

or $4 + (n-1) \times 7 = 151$, $\therefore\ n = 22$

151 is a member of the second sequence.

d $\dfrac{n}{2}(2 \times 120 + (n-1) \times 3) = \dfrac{n}{2}(2 \times 4 + (n-1) \times 7)$

$\therefore\ n(237 + 3n) = n(1 + 7n)$

$\therefore\ 237n + 3n^2 = n + 7n^2$

$\therefore\ 0 = 4n^2 - 236n$

$\therefore\ 0 = 4n(n - 59)$

The sums of the first 59 terms are the same.

e $\dfrac{n}{2}(2 \times 120 + (n-1) \times 3)$

$- \left(\dfrac{n}{2}(2 \times 4 + (n-1) \times 7)\right) = 228$

$\therefore\ n(237 + 3n) - n(1 + 7n) = 2 \times 228$

$\therefore\ 237n + 3n^2 - (n + 7n^2) = 456$

$\therefore\ 0 = 4n^2 - 236n + 456$

$\therefore\ 0 = n^2 - 59n + 114$

$\therefore\ 0 = (n - 57)(n - 2)$

$\therefore\ n = 2$ or 57

The sums of the sequences differ by 228 for both the 2nd and 57th terms.

5 a i $\dfrac{u_2}{u_1} = \dfrac{28}{56} = 0.5$, $\dfrac{u_3}{u_2} = \dfrac{14}{28} = 0.5$

Since $\dfrac{u_2}{u_1} = \dfrac{u_3}{u_2}$, the sequence is geometric.

ii $u_8 = u_1 r^{n-1} = 56 \times 0.5^7 = 0.4375$

iii $S_8 = \dfrac{56 \times (1 - 0.5^8)}{1 - 0.5} = 111.5625$

b i $u_1 r^2 = 24.5$ and $u_1 r^4 = 12.005$

so $r^2 = \dfrac{12.005}{24.5}$

$\therefore\ r = 0.7$

Now $u_1 \times (0.7)^2 = 24.5$

$\therefore\ u_1 = \dfrac{24.5}{(0.7)^2} = 50$

ii General formula is $u_n = 50 \times 0.7^{n-1}$

c $20 \times 0.8^{n-1} = 50 \times 0.7^{n-1}$

$\therefore\ \left(\dfrac{0.8}{0.7}\right)^{n-1} = \dfrac{50}{20} = 2.5$

$\therefore\ n - 1 = \dfrac{\log 2.5}{\log\left(\frac{0.8}{0.7}\right)}$

$\therefore\ n = 7.86$

The first 7 terms of the sequence in **b** are larger than the first seven terms of $u_n = 20 \times 0.8^{(n-1)}$.

d i $S_{30} = \dfrac{20 \times (1 - 0.8^{30})}{1 - 0.8} = 99.9$

ii $S_{50} = 100$

iii $S_{100} = 100$

For large values of n the sum is about 100.

6 a i Interest of 12.5% is added for the year.

Amount owing is $20\,000 \times \dfrac{112.5}{100} = 20\,000 \times 1.125$

Payment of $\$k$ is made.

Amount owing is $(20\,000 \times 1.125) - k$

ii Amount owing at end of second year is

$((20\,000 \times 1.125) - k) \times 1.125 - k$

iii $(20\,000 \times 1.125) \times 1.125 - 1.125k - k = 17\,131.25$

$\therefore \quad 25\,312.5 - 2.125k = 17\,131.25$

$\therefore \quad k = 3850$

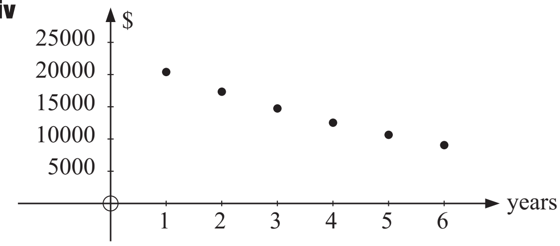
b i Percentage decrease $= \dfrac{24\,000 - 20\,400}{24\,000} \times 100\% = 15\%$

ii Value at the end of the 2nd year $= \$20\,400 \times 0.85$

$= \$17\,340$

iii Value after n years, $t_n = \$24\,000 \times (0.85)^n$

iv



7 a i first month $250 + (5000 \times 1.2\%) = \$310$

ii second month $250 + (4800 \times 1.2\%) = \$307$

iii third month $250 + (4600 \times 1.2\%) = \$304$

b First term is 310. Common difference is -3.

Value of nth payment is $310 + (n-1) \times (-3)$

$= \$(313 - 3n)$

c When $n = 10$, $313 - 3(10) = \$283$

d $\dfrac{5000}{250} = 20$ monthly payments

e $S_{20} = \frac{20}{2}(2 \times 310 + (20-1) \times -3) = \$5630$

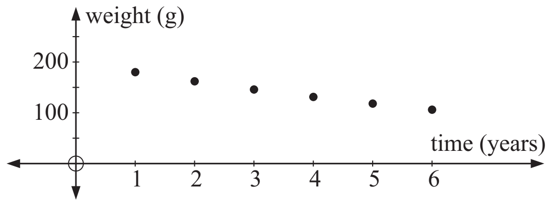
8 a i Weight at start of second year $= 200 \times 0.9 = 180$ g

Weight at start of third year $= 180 \times 0.9 = 162$ g

ii Common ration is 0.9

iii Weight at start of sixth year $= 200 \times 0.9^5 = 118.098$ g

iv



v As $200 \times 0.9^{n-1} = 20$ then $0.9^{n-1} = \frac{20}{200}$

$\therefore \quad n \approx 22.9$ {using a gdc}

The material will weigh less than 20 g at the start of the 23rd year.

b Amount of radioactive material at the beginning of the nth year is $120r^{n-1}$.

At the end of the 6th year, or the beginning of the 7th year, the amount is $120 \times r^{7-1} = 49.152$

$\therefore \quad r^6 = \dfrac{49.152}{120}$

$\therefore \quad r = \left(\dfrac{49.152}{120}\right)^{\frac{1}{6}} = 0.862$

Annual decrease is $1 - 0.862 = 0.138 = 13.8\%$.

SOLUTIONS TO TOPIC 3 SETS, LOGIC AND PROBABILITY

Short questions

1 a $A = \{4, 8, 12, 16\}$, $B = \{2, 4, 6, 8\}$

$A \cup B = \{2, 4, 6, 8, 12, 16\}$

b $A \cap B = \{4, 8\}$

c $C = \{4, 8, 12\}$ or any subset of A that contains 3 elements.

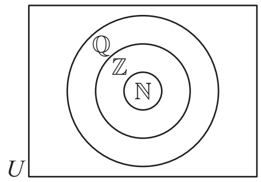
2 a i -6 (or any integer)

ii $3\frac{7}{10}$ (or any number not an integer)

iii 29 (or any positive integer)

iv $\mathbb{Q}' \cap \mathbb{Z} = \varnothing$ and so there are no elements in $\mathbb{Q}' \cap \mathbb{Z}$.

b

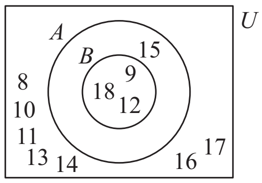


3 $A = \{9, 12, 15, 18\}$, $B = \{9, 12, 18\}$

a $A \cap B = \{9, 12, 18\}$

b $A' = \{8, 10, 11, 13, 14, 16, 17\}$

c



d i $B \subset A$ is true

ii $A' \cap B' = \{8, 10, 11, 13, 14, 16, 17\}$

so $n(A' \cap B') = 7$ is true.

iii Since $B \subset A$, $A \cup B = A$ is true.

4 $F = \{1, 2, 3, 4, 6, 8, 12\}$

$P = \{2, 3, 5, 7, 11, 13, 17, 19\}$

a $n(F) = 7$

b $P \cap F = \{2, 3\}$

c $P \cup F = \{1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 17, 19\}$

d $P' \cap F = \{1, 4, 6, 8, 12\}$

5 $A = \{2, 3, 4, 6, 8, 12, 16, 24, 48\}$

$B = \{6, 12, 18, 24, 30, 36, 42, 48\}$

$C = \{8, 16, 24, 32, 40, 48\}$

a $A \cap B \cap C = \{24, 48\}$

b $n(B) = 8$

c $A' \cap B = \{18, 30, 36, 42\}$

6 a Since b is a prime number less than 10, $b = 2$.

b $\mathbb{Q}$ is the set of even numbers less than 10.

c $a = 9$

7 a i $A \cap B = \varnothing$ is false as A and B have common elements.

ii $A \cap C = C$ is true as C lies entirely within A.

iii $B \subset A'$ is false as A and B have common elements.

iv $C \subset (A \cap B)$ is false as C has no elements in common with $A \cap B$.