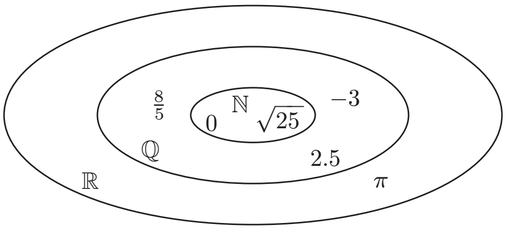
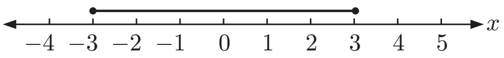
SOLUTIONS TO TOPIC 2 (NUMBER AND ALGEBRA)

SHORT QUESTIONS

1



2 **a**



b

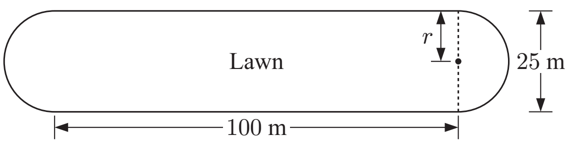
3 **a** $\dfrac{a^2+b}{c} = \dfrac{2.5^2+7}{137} \approx 0.096\,715\,328$

i 0.10 {to 2 decimal places}

ii 0.0967 {to 3 significant figures}

b 9.67×10^{-2}

4



Area = area of two semi-circles + area of rectangle

$= \pi r^2 + 100 \times 2r$

$= \pi \times 12.5^2 + 200 \times 12.5$

a Using $\pi \approx 3$, George's answer is $2968.75\ \text{m}^2$

$\approx 2970\ \text{m}^2$

b The actual area $\approx 2990\ \text{m}^2$

$\therefore$ percentage error $\approx \dfrac{2990-2970}{2990}$

$\approx 0.67\%$ {to 2 significant figures}

5

	ℕ	ℤ	ℚ	ℝ
$(-2)^3$	N	Y	Y	Y
$\sqrt{2}$	N	N	N	Y
0.65	N	N	Y	Y
1.27×10^4	Y	Y	Y	Y

6 $\sqrt{\dfrac{32.76}{3.95 \times 2.63}} \approx 1.775\,806\,022$

a 1.776 {to 3 decimal places}

b 2 {to the nearest integer}

c 1.78 {to 3 significant figures}

d 1.78×10^0 {in standard form}

7 **a** Speed of light

$\approx 186\,280$ miles s^{-1}

$\approx (186\,280 \times 1.609)\ \text{km s}^{-1}$

$\approx (186\,280 \times 1.609 \times 60)\ \text{km min}^{-1}$

$\approx 17\,983\,471\ \text{km min}^{-1}$

$\approx 18\,000\,000\ \text{km min}^{-1}$ {to 3 significant figures}

b $1.80 \times 10^7\ \text{km min}^{-1}$

c The time for light to reach the earth is

$\dfrac{\text{distance}}{\text{speed}} \approx \dfrac{195\,000\,000}{18\,000\,000} \approx 10.8$ minutes.

8 **a** Since the series is arithmetic, $a - 120 = 98 - a$

$\therefore\ 2a = 218$

$\therefore\ a = 109$

If the common difference is d then

$d = 109 - 120 = -11$

$\therefore\ b = 98 - 11$

$= 87$

b The sequence has $u_1 = 120$ and $d = -11$

$\therefore\ u_n = 120 + (n-1)(-11)$

$= 131 - 11n$

c If $131 - 11n = 0$ then $11n = 131$

$\therefore\ n \approx 11.9$

The last term that is positive is u_{11}, so there are 11 positive terms.

9 **a** For interest of 4.5% p.a. the multiplier is 1.045

i After 1 year the value $= £18\,000 \times 1.045$

$= £18\,810$

ii After 2 years the value $= £18\,810 \times 1.045$

$= £19\,656.45$

b After n years the value $= £18\,000 \times 1.045^n$

c After 13 years the value $= £18\,000 \times 1.045^{13}$

$\approx £31\,899.53$

d The investment is trebled when $1.045^n = 3$

$\therefore\ n \approx 24.96$ years

So, the investment will have trebled after 25 years.

10 **a** 5.645 cm ≈ 5.65 cm {to 3 significant figures}

i actual error = 5.65 cm − 5.645 cm

= 0.005 cm

ii percentage error $= \dfrac{\text{actual error}}{\text{exact value}} \times 100\%$

$= \dfrac{0.005}{5.645} \times 100\%$

$\approx 0.0886\%$

b **i** Maximum possible error $= 70 \times 0.032$

$= 2.24\ \text{km h}^{-1}$

ii Minimum possible speed $= (70 - 2.24)\ \text{km h}^{-1}$

$\approx 67.8\ \text{km h}^{-1}$

Maximum possible speed $= (70 + 2.24)\ \text{km h}^{-1}$

$\approx 72.2\ \text{km h}^{-1}$

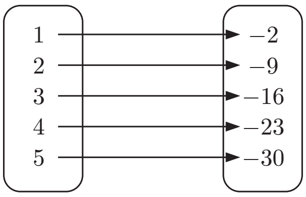
11 **a** $-9 - (-2) = -7$

$-16 - (-9) = -7$

So, this is an arithmetic sequence with common difference -7.

The sequence is $-2, -9, -16, -23, -30, \ldots$

b $m : n \mapsto u_n$



c $u_n = a + (n-1)d$
$= -2 + (n-1)(-7)$
$= 5 - 7n$

12 **a** $u_1 = 1(1+1) = 2, \quad u_2 = 2(2+1) = 6,$
$u_3 = 3(3+1) = 12$

b $u_{15} = 15(15+1) = 240$

c If $u_n = 600$ then $n(n+1) = 600$
$\therefore \; n^2 + n - 600 = 0$
$\therefore \; (n-24)(n+25) = 0$
$\therefore \; n = 24$ or -25
But $n > 0$, so the 24th term is 600.

13

	$\sqrt{4}$	-2	3.75	π	$2.\overline{3}$
$\mathbb{N}$	✓				
$\mathbb{Z}$	✓	✓			
$\mathbb{Q}$	✓	✓	✓		✓
$\mathbb{R}$	✓	✓	✓	✓	✓

14 $2.34 + \dfrac{5.25}{3.10 \times 7.65} \approx 2.561\,378\,874$

a 3 {to the nearest integer}

b 2.5614 {to 4 decimal places}

c 2.56 {to 3 significant figures}

d 2.56×10^0 {in scientific notation}

15 **a** Since the series is arithmetic,
$(k - 166) - (-347) = -185 - (k - 166)$
$\therefore \; k - 166 + 347 = -185 - k + 166$
$\therefore \; 2k = -200$
$\therefore \; k = -100$

b The common difference $d = (k-166) - (-347)$
$= -100 - 166 + 347$
$= 81$
$\therefore$ since the first term $u_1 = -347$,
the general term $u_n = u_1 + (n-1)d$
$= -347 + (n-1)81$
$= -428 + 81n$

c If $-428 + 81n = 0$ then $81n = 428$
$\therefore \; n \approx 5.28$
$\therefore$ the first positive term will be the 6th term, and this term is $u_6 = -428 + 81 \times 6$
$= 58$

16 **a** $u_6 = 49$ so $u_1 + 5d = 49$ .... (1)
$u_{15} = 130$ so $u_1 + 14d = 130$ .... (2)
$\therefore \; 9d = 81$
{subtracting (1) from (2)}
$\therefore$ the common difference $d = 9$

b Since $u_6 = 49$, $u_1 + 5d = 49$
$\therefore \; u_1 + 45 = 49$
$\therefore$ the first term $u_1 = 4$

c The general term of the sequence is
$u_n = u_1 + (n-1)d$
$= 4 + (n-1)9$
$= 9n - 5$
If $9n - 5 = 300$ then $9n = 305$
$\therefore \; n \approx 33.9$
So, the first 33 terms (only) are less than 300.

17 **a** Length of tubing $=$ circumference of circle
$= 2\pi r$
$= \pi d$
$= 4.5\pi$
≈ 14.1 m

b 14 m {to the nearest metre}

c Percentage error $= \dfrac{14 - 4.5\pi}{4.5\pi} \times 100\%$
$\approx -0.970\%$

18 **a** The common ratio $= \dfrac{6.75}{2.25} = 3$

b Since $u_1 = 0.75$, the general term $u_n = 0.75 \times 3^{n-1}$

c $S_n = \dfrac{u_1(r^n - 1)}{r - 1}$
$\therefore \; S_{10} = \dfrac{0.75(3^{10} - 1)}{3 - 1} = 22\,143$

19 **a** For an arithmetic series, $S_n = \dfrac{n}{2}(2u_1 + (n-1)d)$
$\therefore \; \frac{7}{2}(2u_1 + 6 \times 14) = 329$
$\therefore \; 7u_1 + 294 = 329$
$\therefore \; 7u_1 = 35$
$\therefore \; u_1 = 5$

b If $S_n = 69\,800$ then
$\dfrac{n}{2}(2u_1 + (n-1)d) = 69\,800$
$\therefore \; \dfrac{n}{2}(10 + (n-1)14) = 69\,800$
Using technology, $n = 100$

20 **a** time taken $= \dfrac{\text{distance travelled}}{\text{average speed}}$
$= \dfrac{80 \text{ km}}{45 \text{ km h}^{-1}}$
≈ 1.7778 hours
≈ 107 minutes

b time in mountains $\approx (3 - 1.7778)$ hours
≈ 1.2222 hours
distance $=$ average speed $\times$ time
$\approx 25 \text{ km h}^{-1} \times 1.2222$ hours
≈ 30.6 km

21 **a** Since the series is arithmetic,
$x - 27 = 42 - x$
$\therefore \; 2x = 69$
$\therefore \; x = 34.5$
The common difference $d = x - 27 = 7.5$

b $u_n = u_1 + (n-1)d$

$\therefore$ since $u_5 = 27$, $27 = u_1 + 4d$

$\therefore$ $u_1 = 27 - 4 \times 7.5$

$\therefore$ $u_1 = -3$

c $u_n = -3 + (n-1)7.5$

$= 7.5n - 10.5$

If $7.5n - 10.5 = 2400$

then $7.5n = 2410.5$

$\therefore$ $n = 321.4$

So, the first term greater than 2400 is the 322nd term, and this is $u_{322} = -3 + 321 \times 7.5$

$= 2404.5$

22 **a** Since the sequence is geometric,

$$\frac{x}{45} = \frac{281.25}{x}$$

$\therefore$ $x^2 = 45 \times 281.25$

$\therefore$ $x = \sqrt{12\,656.25}$ {since $x > 0$}

$\therefore$ $x = 112.5$

So, the common ratio $r = \dfrac{112.5}{45} = 2.5$

b For a geometric series, $S_n = \dfrac{u_1(r^n - 1)}{r - 1}$

So, $u_5 + u_6 + ... + u_{12}$

$= S_{12} - S_4$

$= \dfrac{45(2.5^{12} - 1)}{11} - \dfrac{45(2.5^4 - 1)}{3}$

$\approx 243\,000$

23 **a** Error $= 4$ m $- 3.94$ m

$= 0.06$ m

b Length of the joined pipes $= 5 \times 3.94$ m

$= 19.7$ m

c The approximated length would be 5×4 m $= 20$ m

$\therefore$ the error $= 20$ m $- 19.7$ m

$= 0.3$ m

d The percentage error $= \dfrac{0.3}{19.7} \times 100\%$

$\approx 1.52\%$

24 **a** $\dfrac{4.5}{5} = \dfrac{9}{10}$ and $\dfrac{4.05}{4.5} = \dfrac{9}{10}$

$\therefore$ the sequence is geometric with common ratio $\frac{9}{10}$.

b The third bounce has height $4.05 \times \frac{9}{10} = 3.645$ m

≈ 3.65 m

c The total distance travelled

$= 5 + 2 \times 4.5 + 2 \times 4.05 + 2 \times 3.645$

≈ 29.4 m

25 **a** **i** $A = \{-1, 0, 1, 2\}$ **ii** $B = \{2, 3, 5, 7, 11, 13\}$

iii $C = \{-\sqrt{8}, +\sqrt{8}\}$

b **i** false **ii** true **iii** true

26 **a** There are $\frac{1}{2} \times 128 = 64$ clubs in the second round.

There are $\frac{1}{2} \times 64 = 32$ clubs in the third round.

b Let u_n be the number of teams in the nth round.

The sequence is geometric with $u_1 = 128$ and $r = \frac{1}{2}$

$\therefore$ $u_n = 128 \times \left(\frac{1}{2}\right)^{n-1}$

c If $u_n = 1$ then

$128 \times \left(\frac{1}{2}\right)^{n-1} = 1$

$2^{n-1} = 128$

$\therefore$ $2^{n-1} = 2^7$

$\therefore$ $n - 1 = 7$ {equating indices}

$\therefore$ $n = 8$

$\therefore$ in the 8th round there would be only 1 team, so it takes 7 rounds to determine the winner.

27 **a** Let the price of a book be b NZD and the price of a pen be p NZD.

So, $15b + 8p = 209$ .... (1)

and $7b + 3p = 96.25$ .... (2)

b Solving (1) and (2) simultaneously using technology, $b = 13$ and $p = 1.75$.

c The total cost of 10 books and 6 pens

$= 10 \times 13 + 6 \times 1.75$

$= 140.50$ NZD

28 **a** The series is geometric with $u_2 = 14.5$, and $u_5 = 1.8125$.

Now $u_2 = u_1 \times r$ and $u_5 = u_1 \times r^4$

$\therefore$ $\dfrac{u_5}{u_2} = \dfrac{u_1 r^4}{u_1 r} = r^3$

$\therefore$ $r^3 = \dfrac{1.8125}{14.5} = 0.125$

$\therefore$ $r = 0.5$

The common ratio is 0.5.

b $u_2 = u_1 \times r$

$\therefore$ $u_1 = \dfrac{u_2}{r} = \dfrac{14.5}{0.5} = 29$

$\therefore$ the first term is 29.

c For a geometric series, $S_n = \dfrac{u_1(1 - r^n)}{1 - r}$

$\therefore$ $S_5 = \dfrac{29(1 - 0.5^5)}{1 - 0.5}$

$\therefore$ $S_5 = 56.1875$

29 **a** Each year the population is multiplied by 1.09

$\therefore$ the population in 2010 is $1200 \times 1.09^5 \approx 1850$ people.

b If n is the number of years after 2005, then

$1200 \times 1.09^n = 2500$

$\therefore$ $1.09^n = \frac{25}{12}$

Using technology, $n \approx 8.52$

$\therefore$ the population reaches 2500 during the year 2014.

c If the population increased with multiplier r each year, then $1200 \times r^{10} = 3200$

$\therefore$ $r^{10} = \frac{32}{12} = \frac{8}{3}$

Using technology, $r \approx 1.103$

$\therefore$ the rate of increase $\approx 10.3\%$

30 **a** The series is geometric if $\dfrac{k-6}{k} = \dfrac{k}{2k-9}$

$\therefore$ $(k-6)(2k-9) = k^2$

$\therefore$ $2k^2 - 9k - 12k + 54 = k^2$

$\therefore$ $k^2 - 21k + 54 = 0$

$\therefore$ $(k-3)(k-18) = 0$

$\therefore$ $k = 3$ or 18

b The sequence is arithmetic if

$$(k-6)-k = k-(2k-9)$$
$$\therefore \quad -6 = -k+9$$
$$\therefore \quad k = 15$$

31 **a** **i** $s=16,\ r=5$ **ii** $u_1=27,\ d=5$

b $a+12p=20$ .... (1)

$a+22p=34$ .... (2)

Using technology, $a=3.20$ and $p=1.40$.

32 **a**
$$178-4n = 7n+57$$
$$\therefore \quad 11n = 121$$
$$\therefore \quad n = 11$$

So, the sequences have the 11th term in common, and this term is $178-4\times 11 = 134$.

b $C=R$ if $25n = 21\,000+7.5n$
$$\therefore \quad 17.5n = 21\,000$$
$$\therefore \quad n = 1200$$

So, the cost equals the revenue when 1200 goods are produced and sold.

c If $\dfrac{n(n+1)}{2}=435$

then $n(n+1)=870$
$$\therefore \quad n^2+n-870=0$$
$$\therefore \quad (n+30)(n-29)=0$$
$$\therefore \quad n=29 \quad \{\text{since } n>0\}$$

So, the sum exceeds 435 for $n>29$.

33 **a** $u_7 = 4374\left(\frac{1}{3}\right)^6$
$= 6$

b If $4374\left(\frac{1}{3}\right)^{n-1}=0.1$

then $3^{n-1}=43\,740$

Using technology, $n\approx 10.7$

So, $u_n \leqslant 0.1$ for $n\geqslant 11$.

c For a geometric series, $S_n=\dfrac{u_1(1-r^n)}{1-r}$
$$\therefore \quad S_8 = \frac{4374\left(1-\left(\frac{1}{3}\right)^8\right)}{1-\frac{1}{3}}$$
$$\therefore \quad S_8 = 6560$$

34 **a** Let the length be l cm $\therefore$ $2x+2l=80$
$$\therefore \quad 2l = 80-2x$$
$$\therefore \quad l = 40-x$$

b The area $A = l\times x$
$= x(40-x)$
$= 40x-x^2$ cm^2

c
$$40x-x^2=375$$
$$\therefore \quad x^2-40x+375=0$$
$$\therefore \quad (x-15)(x-25)=0$$
$$\therefore \quad x=15 \text{ or } 25$$
$$\therefore \quad l=25 \text{ or } 15$$

So, the rectangle is 15 cm $\times$ 25 cm.

35 **a** $s=0$ where $ut-5t^2=0$
$$\therefore \quad 70t-5t^2=0$$
$$\therefore \quad -5t(t-14)=0$$
$$\therefore \quad t=0 \text{ or } 14$$

The rocket is in the air for 14 seconds.

b $s=30$ when $ut-5t^2=30$
$$\therefore \quad 70t-5t^2=30$$
$$\therefore \quad 5t^2-70t+30=0$$

Using technology, $t\approx 0.443$ or 13.557

The rocket is above 30 m for

$13.557-0.443\approx 13.1$ seconds

36 $23r+b=19.15$ .... (1)

$15r+b=15.95$ .... (2)

Solving (1) and (2) simultaneously using technology, $r=0.4$ and $b=9.95$.

37 **a** Since $x=3$ and $x=-5$ are solutions,
$$(x^2+bx+c) = (x-3)(x+5)$$
$$= (x^2+2x-15)$$
$$\therefore \quad b=2 \text{ and } c=-15$$

b
$$f(0) = a(0+0+c)$$
$$= ac$$
$$\therefore \quad ac = 30$$
$$\therefore \quad -15a = 30$$
$$\therefore \quad a = -2$$

LONG QUESTIONS

1 **a** **i** $u_1 = 4+11\times 1 = 15$

$u_2 = 4+11\times 2 = 26$

ii For an arithmetic sequence,
$$S_n = \frac{n}{2}(2u_1+(n-1)d)$$

In this case $u_1=15$ and $d=11$
$$\therefore \quad S_{10} = \tfrac{10}{2}(30+9\times 11)$$
$$= 645$$

b **i** $u_5 = 4(2.2)^4 = 93.7024$ {exact}

ii For a geometric sequence, $S_n = \dfrac{u_1(r^n-1)}{r-1}$

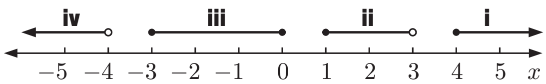
In this case $u_1=4$ and $r=2.2$
$$\therefore \quad S_{10} = \frac{4(2.2^{10}-1)}{2.2-1} \approx 8850$$

c For the arithmetic sequence: $u_1=15,\ u_2=26,\ u_3=37,\ u_4=48,\ u_5=59$

For the geometric sequence: $u_1=4,\ u_2=8.8,\ u_3=19.36,\ u_4=42.592,\ u_5=93.7024$

$\therefore$ the fifth term of geometric sequence is greater than that of the arithmetic sequence. The difference between these terms is ≈ 34.7.

2 **a**



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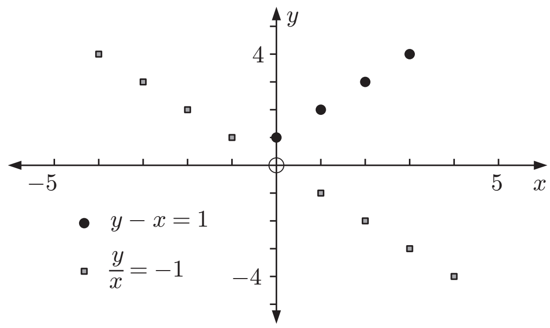


c Let $x=-2,\ y=-3$, so $x^2=4$ and $y^2=9$.

We see that $x>y$ but $x^2<y^2$.

3 **a**
$$u_5 = u_1 + 4d$$
$$\therefore\ u_1 + 4d = 51 \ \text{.... (1)}$$
$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$
$$\therefore\ nu_1 + \frac{n(n-1)}{2}d = S_n$$
$$\therefore\ 5u_1 + 10d = 185 \quad \{S_5 = 185\}$$
$$\therefore\ u_1 + 2d = 37 \ \text{.... (2)}$$

b Using technology, $d = 7$ and $u_1 = 23$

c $u_n = 23 + (n-1)7$
$= 7n + 16$

If $7n + 16 = 1000$ then $7n = 984$
$$\therefore\ n \approx 140.6$$
$\therefore$ the first term to exceed 1000 is the 141st term, and this is $u_{141} = 7 \times 141 + 16$
$= 1003$

d **i** $S_n = \frac{n}{2}(2u_1 + (n-1)d)$
$$\therefore\ \frac{k}{2}(2 \times 23 + (k-1)7) = 3735$$
$$\therefore\ k(46 + 7k - 7) = 7470$$
$$\therefore\ 7k^2 + 39k - 7470 = 0 \ \text{ as required.}$$

ii Using technology, $k = -\frac{249}{7}$ or 30
But $k > 0$, so $k = 30$.

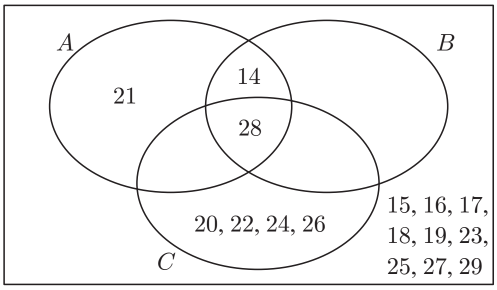
4 **a** **i** false since '2' is not included in $\{x \mid x < 2,\ x \in \mathbb{R}\}$

ii true

iii false since $2^2 + 2 = 6 \neq 2$

b **i** $A = \{14, 21, 28\}$, $B = \{14, 28\}$,
$C = \{20, 22, 24, 26, 28\}$.

ii



c **ii** is false. Consider $p = 1$, $q = 2$. $1 - 2 \neq 2 - 1$

5 **a** $25.32 \times \dfrac{6.057}{2.4 \times \sqrt{5.14}} \approx 28.185\,68$

i 28.186 {to 5 significant figures}

ii 28.2 {to the nearest tenth}

iii 30 {to 1 significant figure}

b **i** The actual length covered is $3 \times 3.63 = 10.89$ m.

ii The error between the actual length and the stated length is $10.80 - 10.89 = -0.09$ m.

Percentage error $= \dfrac{10.80 - 10.89}{10.89} \times 100\%$
$= -0.826\%$

c **i** The minimum volume needed is
$10.89 \times 10.89 \times 0.095 \approx 11.266 \text{ m}^3 \approx 11.3 \text{ m}^3$
Maximum volume needed is
$10.89 \times 10.89 \times 0.105 \approx 12.452 \text{ m}^3 \approx 12.5 \text{ m}^3$

ii Planned volume $= 10.8 \times 10.8 \times 0.1$
$= 11.664 \text{ m}^3$

The difference in cost is
$(12.452 - 11.664) \times 47.50$
$=$ €37.43

iii The percentage difference in cost is
$$\frac{37.43}{11.664 \times 47.50} \times 100\% = 6.76\%$$

6 **a** **i** 120, 123, 126 **ii** 4, 11, 18

b The second sequence $u_{n+1} = u_n + 7$ is arithmetic with $u_1 = 4$ and common difference $d = 7$
$$\therefore\ u_n = 4 + 7(n-1)$$
The sequences have a term in common when
$$120 + 3(n-1) = 4 + 7(n-1)$$
$$\therefore\ 117 + 3n = -3 + 7n$$
$$\therefore\ 4n = 120$$
$$\therefore\ n = 30$$
So, the 30th terms are the same, and they have value
$$u_{30} = 4 + 7 \times 29$$
$$= 207$$

c If $120 + 3(n-1) = 151$
then $3n + 117 = 151$
$\therefore\ 3n = 34$
$\therefore\ n \approx 11.3$ which is not an integer.

If $4 + 7(n-1) = 151$
then $7n - 3 = 151$
$\therefore\ 7n = 154$
$\therefore\ n = 22$

So, 151 is the 22nd term of the second sequence.

d For an arithmetic sequence,
$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$
$$\therefore\ \frac{n}{2}(2 \times 120 + (n-1)3) = \frac{n}{2}(2 \times 4 + (n-1)7)$$
$$\therefore\ \frac{n}{2}(240 + 3n - 3 - 8 - 7n + 7) = 0$$
$$\therefore\ \frac{n}{2}(-4n + 236) = 0$$
$$\therefore\ -2n(n - 59) = 0$$
$$\therefore\ n = 59$$
$$\{\text{since } n > 0\}$$
The sums of the first 59 terms are the same.

e $\frac{n}{2}(2 \times 120 + (n-1)3)$
$$-\frac{n}{2}(2 \times 4 + (n-1)7) = 228$$
$$\therefore\ \frac{n}{2}(-4n + 236) - 228 = 0$$
$$\therefore\ -2n^2 + 118n - 228 = 0$$
$$\therefore\ n^2 - 59n + 114 = 0$$
$$\therefore\ (n-57)(n-2) = 0$$
$$\therefore\ n = 2 \text{ or } 57$$
The sum of the first sequence exceeds the sum of the second by 228 for both the 2nd and 57th terms.

7 **a** Total surface area = length × width
$= (2x + 8)(x - 2) \text{ m}^2$
$= 2x^2 - 4x + 8x - 16 \text{ m}^2$
$= 2x^2 + 4x - 16 \text{ m}^2$

b
$$2x^2 + 4x - 16 = 224$$
$$\therefore\ 2x^2 + 4x - 240 = 0$$
$$\therefore\ x^2 + 2x - 120 = 0$$
$$\therefore\ (x + 12)(x - 10) = 0$$
$$\therefore\ x = -12 \text{ or } 10$$
But $x > 0$, so $x = 10$

c Total capacity

$= \text{capacity of swimming pool} + \text{capacity of wading pool}$
$= 1.8 \times (x-2) \times (x+8) + 0.5 \times (x-2) \times x$ kL
$= 1.8 \times 8 \times 18 + 0.5 \times 8 \times 10$ kL
$= 299.2$ kL
$= 299\,200$ L

d **i** Original capacity of wading pool
$= 0.5 \times 8 \times 10$ kL
$= 40$ kL
$= 40\,000$ L
$\therefore$ new capacity of wading pool
$= 40\,000 - 2400$
$= 37\,600$ L

ii Let the new depth of the wading pool be d m.
$\therefore\ d \times 8 \times 10 = 37.6$
$\therefore\ d = 0.47$ m
So, the new depth of the wading pool is 47 cm.

8 **a** **i** $\dfrac{u_2}{u_1} = \dfrac{28}{56} = 0.5, \qquad \dfrac{u_3}{u_2} = \dfrac{14}{28} = 0.5$
Since $\dfrac{u_2}{u_1} = \dfrac{u_3}{u_2}$, the sequence is geometric.

ii $u_n = u_1 r^{n-1}$
$\therefore\ u_8 = 56 \times 0.5^7 = 0.4375$

iii $S_n = \dfrac{u_1(1-r^n)}{1-r}$
$\therefore\ S_8 = \dfrac{56 \times (1-0.5^8)}{1-0.5} = 111.5625$

b **i** $u_3 = u_1 r^2 = 24.5$
$u_5 = u_1 r^4 = 12.005$
$\therefore\ \dfrac{u_5}{u_3} = r^2 = \dfrac{12.005}{24.5} = 0.49$
$\therefore\ r = \pm 0.7$
But all of the terms are positive, so $r > 0$
$\therefore\ r = 0.7$
Now, $u_1 \times 0.49 = 24.5$
$\therefore\ u_1 = 50$
So, the first term is 50 and the common ratio is 0.7.

ii The general term $u_n = 50 \times 0.7^{n-1}$

c If $20 \times 0.8^{n-1} = 50 \times 0.7^{n-1}$
then $\left(\frac{0.8}{0.7}\right)^{n-1} = \frac{50}{20} = 2.5$
Using technology, $n \approx 7.86$
So, the first 7 terms of the sequence in **b** are larger than the first 7 terms of $u_n = 20 \times 0.8^{n-1}$, $\therefore\ n = 7$.

d $S_n = \dfrac{u_1 \times (1-r^n)}{1-r}$

i $S_{30} = \dfrac{20 \times (1-0.8^{30})}{1-0.8} \approx 99.9$

ii $S_{50} = \dfrac{20 \times (1-0.8^{50})}{1-0.8} \approx 100$

iii $S_{100} = \dfrac{20 \times (1-0.8^{100})}{1-0.8} \approx 100$

For large values of n, the sum approaches 100.

9 **a** **i** At the end of the year, 12.5% interest is added, and then \$$k$ is paid off.
12.5% interest has multiplier 1.125, so the amount owing is $\$(20\,000 \times 1.125 - k)$.

ii At the end of the second year, the amount owing is $\$((20\,000 \times 1.125 - k) \times 1.125 - k)$.

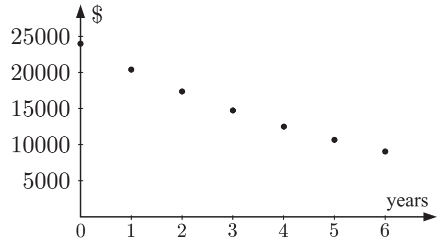
iii Using the expression in **ii**,
$25\,312.5 - 1.125k - k = 17\,131.25$
$\therefore\ 2.125k = 8181.25$
$\therefore\ k = 3850$

b **i** $\text{percentage decrease} = \dfrac{\text{decrease}}{\text{original value}} \times 100\%$
$= \dfrac{24\,000 - 20\,400}{24\,000} \times 100\%$
$= 15\%$

ii Depreciating at 15%, the value after two years
$= \$20\,400 \times 0.85$
$= \$17\,340$

iii The value after n years $= \$24\,000 \times (0.85)^n$

iv



10 **a** **i** $A_1 = 6$ **ii** Common difference $d = 19$

b **i** $G_n = G_1 \times r^{n-1}$
$\therefore\ G_3 = G_1 \times r^2 = 1920$
and $G_{10} = G_1 \times r^9 = 15$
$\therefore\ \dfrac{G_{10}}{G_3} = r^7 = \frac{15}{1920}$
$\therefore\ r^7 = \frac{1}{128} = \left(\frac{1}{2}\right)^7$
$\therefore\ r = \frac{1}{2}$
The common ratio is $\frac{1}{2}$.

ii $G_1 \times \left(\frac{1}{2}\right)^2 = 1920$
$\therefore\ G_1 = 7680$

c If $A_n = G_n$ then $19n - 13 = 7680 \times \left(\frac{1}{2}\right)^{n-1}$
Using technology, $n = 7$
The 7th term is common, and this is $A_7 = G_7 = 120$

d **i** A_n is arithmetic, so $S_{A_n} = \dfrac{n}{2}(2A_1 + (n-1)d)$
$= \dfrac{n}{2}(12 + (n-1)19)$
$= \frac{19}{2}n^2 - \frac{7}{2}n$

ii G_n is geometric, so $S_{G_n} = \dfrac{G_1(1-r^n)}{1-r}$
$= \dfrac{7680(1-(\frac{1}{2})^n)}{1-\frac{1}{2}}$
$= 15\,360(1-(\frac{1}{2})^n)$

iii If $\frac{19}{2}n^2 - \frac{7}{2}n = 15\,360\left(1-(\frac{1}{2})^n\right)$
Using technology, $n = 0$ or $n \approx 40.4$
We know $n > 0$, so $S_{A_n} < S_{G_n}$ for $1 \leqslant n \leqslant 40$
It takes 41 terms for S_{A_n} to become greater than S_{G_n}.

11 **a** **i** $\text{Interest} = \$5000 \times 0.012 = \$60$
$\therefore\ \text{repayment} = \$250 + \$60 = \$310$

ii $\text{New principal} = \$5000 + \$60 - \$310$
$= \$4750$
$\text{Interest} = \$4750 \times 0.012 = \$57$
$\therefore\ \text{repayment} = \$250 + \$57 = \$307$

iii New principal $= \$4750 + \$57 - \$307$
$= \$4500$
Interest $= \$4500 \times 0.012 = \$54$
$\therefore$ repayment $= \$250 + \$54 = \$304$

b First term $u_1 = \$310$
Common difference $d = -\$3$
The nth repayment $u_n = \$(310 - 3(n-1))$
$= \$(313 - 3n)$

c $u_{10} = \$(313 - 3 \times 10)$
$= \$283$

d Every month the principal reduces by \$250
$\therefore$ it will take $\frac{5000}{250} = 20$ monthly repayments.

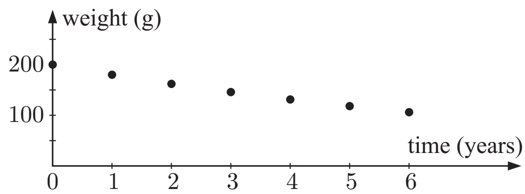
e For the arithmetic sequence, $S_n = \dfrac{n}{2}(2u_1 + (n-1)d)$
$\therefore$ the total amount repaid is
$S_{20} = \$\frac{20}{2}(2 \times 310 + 19 \times (-3))$
$= \$10(620 - 57)$
$= \$5630$

12 **a** **i** Weight at start of second year
$= 200 \times 0.9 = 180$ g
Weight at start of third year
$= 180 \times 0.9 = 162$ g

ii Common ratio is 0.9

iii Weight at start of sixth year
$= 200 \times 0.9^5 = 118.098$ g

iv



v Weight at start of nth year $= 200 \times 0.9^{n-1}$
If $200 \times 0.9^{n-1} = 20$ then $0.9^{n-1} = \frac{1}{10}$
Using technology, $n \approx 22.9$
So, the material will weigh less than 20 g at the start of the 23rd year.

b The amount of radioactive material at the beginning of the nth year is $120r^{n-1}$.
At the end of the 6th year, or the beginning of the 7th year, the amount is
$120 \times r^{7-1} = 49.152$
$\therefore \quad r^6 = \dfrac{49.152}{120}$
$\therefore \quad r = \left(\dfrac{49.152}{120}\right)^{\frac{1}{6}} \approx 0.862$
The annual decrease is
$\approx 1 - 0.862 \approx 0.138 \approx 13.8\%$.

SOLUTIONS TO TOPIC 3 (SETS, LOGIC AND PROBABILITY)

SHORT QUESTIONS

1 $A = \{0, 3, 6, 9\}, \quad B = \{1, 2, 5, 10\}$

a $A \cap B = \varnothing$

b $A \cup B = \{0, 1, 2, 3, 5, 6, 9, 10\}$

c $(A \cup B)' = \{4, 7, 8, 11\}$

2 **a** $A = \{4, 8, 12, 16\}, \; B = \{2, 4, 6, 8\}$
$A \cup B = \{2, 4, 6, 8, 12, 16\}$

b $A \cap B = \{4, 8\}$

c $C = \{4, 8, 12\}$ (other answers are possible)

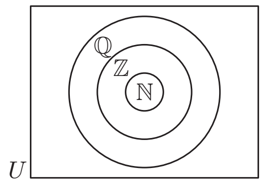
3 **a** **i** -6 (or any integer)

ii $3\frac{7}{10}$ (or any real number not an integer)

iii 29 (or any non-negative integer)

iv $\mathbb{Q}' \cap \mathbb{Z} = \varnothing$ so there are no elements in $\mathbb{Q}' \cap \mathbb{Z}$.

b

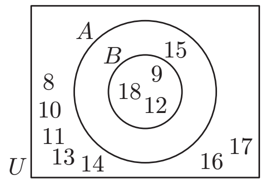


4 $A = \{9, 12, 15, 18\}, \; B = \{9, 12, 18\}$

a $A \cap B = \{9, 12, 18\}$

b $A' = \{8, 10, 11, 13, 14, 16, 17\}$

c



d **i** $B \subset A$ is true

ii $A' \cap B' = \{8, 10, 11, 13, 14, 16, 17\}$
so $n(A' \cap B') = 7$ is true.

iii Since $B \subset A$, $A \cup B = A$ is true.

5 $F = \{1, 2, 3, 4, 6, 8, 12\}$
$P = \{2, 3, 5, 7, 11, 13, 17, 19\}$

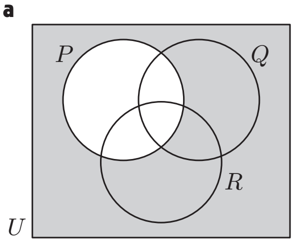
a $n(F) = 7$

b $P \cap F = \{2, 3\}$

c $P \cup F = \{1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 17, 19\}$

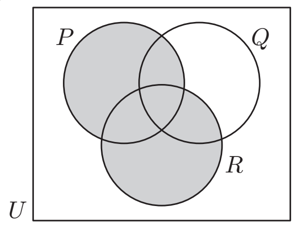
d $P' \cap F = \{1, 4, 6, 8, 12\}$

6 **a**



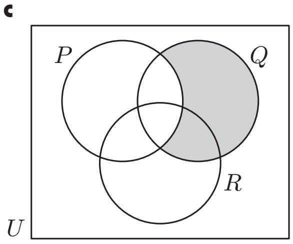
P'

b



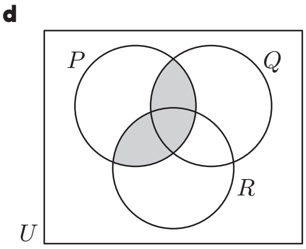
$P \cup R$

c



$P' \cap Q$

d



$P \cap (Q \cup R)$

7 **a** **i** $25 + y + z$ students **ii** x students

iii $13 + x + y + z$ students