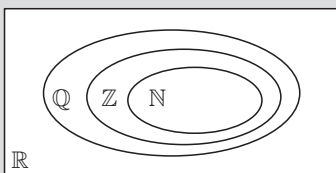


TOPIC 2: NUMBER AND ALGEBRA

MATHEMATICAL LANGUAGE

You should understand these mathematical terms, and be able to use them appropriately:

- addition, subtraction, multiplication, division
- sum, difference, product, quotient
- index or exponent
- prime, composite
- factors, prime factors, common factors, highest common factor (HCF)
- multiples, lowest common multiple (LCM)
- natural numbers $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$
- integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- rational numbers $\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$
- real numbers $\mathbb{R} = \{\text{all numbers on the number line}\}$



THE FUNDAMENTAL THEOREM OF ARITHMETIC

Every composite number can be written as the product of prime factors in exactly one way (ignoring order).

ORDER OF OPERATIONS

Brackets

Exponents

Divisions

Multiplications

} as you come to them

Additions

Subtractions

} as you come to them

SCIENTIFIC NOTATION (STANDARD FORM)

$$a \times 10^k \text{ where } 1 \leq a < 10 \text{ and } k \in \mathbb{Z}$$

- For example:
- 2565 is scientific notation is 2.565×10^3
 - 3.04×10^{-3} as a decimal is 0.003 04

INTERNATIONAL SYSTEM OF UNITS (SI UNITS)

You should understand and be able to use SI units, in particular those for distance, mass, temperature, and time.

You should be able to:

- perform metric conversions $5.3 \text{ kg} = 5300 \text{ g}$
- use areas and volumes $8 \text{ m} \times 5 \text{ m} = 40 \text{ m}^2$
 $3 \text{ m} \times 2 \text{ m} \times 2 \text{ m} = 12 \text{ m}^3$
- use formulae to convert between temperature units
- use rates where we divide units

$$\text{speed (m s}^{-1}\text{)} = \frac{\text{distance (m)}}{\text{time (s)}}$$

APPROXIMATION AND ESTIMATION

Rounding numbers:

- if the digit after the one being rounded is less than 5, round down
- if the digit after the one being rounded is 5 or more, round up.

Significant figures are counted from the first non-zero digit from the left.

For example:

$$3.413 \approx 3.41 \quad (\text{to 2 decimal places})$$

$$0.034\ 56 \approx 0.0346 \quad (\text{to 3 significant figures})$$

$$236.5 \approx 237 \quad (\text{to the nearest whole number})$$

A **measurement** is accurate to $\pm \frac{1}{2}$ of the smallest division on the scale.

An **approximation** is a value given to a number which is close to, but not equal to, its true value.

An **estimation** is a value which is found by judgement or prediction instead of carrying out a more accurate measurement.

If the exact value is V_E and the approximate value is V_A then:

- **error** = $V_A - V_E$
- **percentage error** $E = \frac{V_A - V_E}{V_E} \times 100\%$

We usually state percentage error as the **absolute percentage error** which is the size of the percentage error, ignoring its sign.

LAWS OF ALGEBRA

Index or Exponent Laws	
$a^m \times a^n = a^{m+n}$	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$
$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$	$a^0 = 1, a \neq 0$
$(a^m)^n = a^{m \times n}$	$a^{-x} = \frac{1}{a^x}$ and $\frac{1}{a^{-x}} = a^x$
$(ab)^n = a^n b^n$	

Expansion and Factorisation Laws	
Distributive law	$a(b+c) = ab+ac$
FOIL rule	$(a+b)(c+d) = ac+ad+bc+bd$
Difference of two squares	$(a+b)(a-b) = a^2 - b^2$
Perfect squares	$(a+b)^2 = a^2 + 2ab + b^2$ $(a-b)^2 = a^2 - 2ab + b^2$

EQUATIONS AND FORMULAE

You should understand the meanings of: expression, equation, inequality, formula, substitution, evaluate.

Linear equations can be written in the form $ax+b=0$ where x is the **variable** and a, b are constants.

You should be able to solve linear equations:

- using inverse operations
- using technology.

You should be able to rearrange formulae to make another variable the subject.

Linear simultaneous equations occur when there are two equations in two unknowns. You should be able to solve linear simultaneous equations:

- by substitution
- by elimination
- using technology

Exponential equations are equations where the variable appears in an index or exponent.

If we can write the equation so the base number is the same on each side, we can then solve by **equating indices**.

$$\text{If } a^x = a^k \text{ then } x = k.$$

If we cannot write the equation so the base number is the same on each side, we should solve using technology.

QUADRATIC FACTORISATION

- take out any common factors
- look for special cases:
 - ▶ difference of two squares $x^2 - a^2 = (x + a)(x - a)$
 - ▶ perfect squares $x^2 + 2ax + a^2 = (x + a)^2$
 $x^2 - 2ax + a^2 = (x - a)^2$

- look for sum and product type
 $x^2 + (p + q)x + pq = (x + p)(x + q)$

- 'splitting' the x -term

For $ax^2 + bx + c$

- ▶ find ac
- ▶ look for two factors of ac which add to b
- ▶ if the factors are p and q , replace bx by $px + qx$
- ▶ complete the factorisation.

QUADRATIC EQUATIONS

Quadratic equations have the form $ax^2 + bx + c = 0$ where $a \neq 0$.

The solutions of the equation are called its **roots**.

These values for x are also called the **zeros** of the quadratic expression $ax^2 + bx + c$.

$$\text{If } x^2 = k \text{ then } \begin{cases} x = \pm\sqrt{k} & \text{if } k > 0 \\ x = 0 & \text{if } k = 0 \\ \text{there are no real solutions} & \text{if } k < 0. \end{cases}$$

You should be able to solve quadratic equations:

- using factorisation and the Null Factor law
- using technology.

SEQUENCES

A **number sequence** is a list of numbers which follow a pattern.

The numbers in a sequence are called its **members** or **terms**.

We can define a sequence by:

- listing terms
- using words
- a formula for the **general term** or **n th term**.

An **arithmetic sequence** is a sequence in which each term differs from the previous one by the same fixed number.

$\{u_n\}$ is arithmetic $\Leftrightarrow u_{n+1} - u_n = d$ for all $n \in \mathbb{Z}^+$, where d is a constant called the **common difference**.

The general term of an arithmetic sequence is $u_n = u_1 + (n - 1)d$.

An application of arithmetic sequences is **simple interest**.

A sequence is **geometric** if each term can be obtained from the previous one by multiplying by the same non-zero constant.

$\{u_n\}$ is geometric $\Leftrightarrow \frac{u_{n+1}}{u_n} = r$ for all $n \in \mathbb{Z}^+$,

where r is a constant called the **common ratio**.

The general term of a geometric sequence is $u_n = u_1 r^{n-1}$.

Applications of geometric sequences include **compound interest**, and **growth** and **decay** problems.

SERIES

A **series** is the addition of the terms of a sequence.

For the sequence $\{u_n\}$, the series including the first n terms is $u_1 + u_2 + u_3 + \dots + u_n$.

The result of this addition is $S_n = u_1 + u_2 + u_3 + \dots + u_n$.

For an **arithmetic series**:

$$S_n = \frac{n}{2}(u_1 + u_n) \quad \text{or} \quad S_n = \frac{n}{2}(2u_1 + (n - 1)d)$$

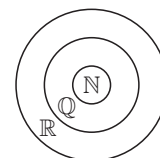
For a **geometric series**:

$$S_n = \frac{u_1(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{u_1(1 - r^n)}{1 - r}$$

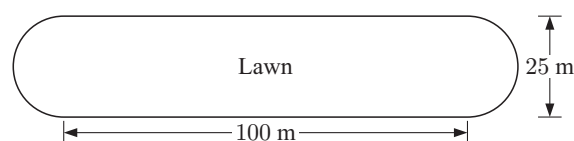
SKILL BUILDER - SHORT QUESTIONS

- 1 Place the following numbers in the appropriate region of the Venn diagram.

$$2.5, \pi, -3, \frac{8}{5}, 0, \sqrt{25}$$



- 2 a Draw a number line to represent the set $X = \{x \mid -3 \leq x \leq 3, x \in \mathbb{R}\}$
- b Use a number line to clearly represent each of the following:
- i $x < -1$
 - ii $0 \leq x < 2$
 - iii $x \geq 2$
- 3 a If $a = 2.5$, $b = 7$ and $c = 137$, calculate $\frac{a^2 + b}{c}$.
Give your answer:
- i correct to 2 decimal places
 - ii correct to 3 significant figures.
- b Write your answer from a ii in scientific notation.
- 4 Gordon needs to fertilise the large garden landscape shown in the diagram:



The ends of this garden landscape are semi-circular, and the middle is rectangular.

Gordon calculates the area of the garden using correct formulae, but approximates the value of π to one significant figure, so $\pi \approx 3$.

- Write down Gordon's approximation for the area of the garden landscape.
 - Calculate the percentage error in Gordon's approximation. Round your answer to 2 significant figures.
- 5 In the table below, indicate using a Y (yes) or N (no) which number sets the following numbers are members of:

	N	Z	Q	R
$(-2)^3$				Y
$\sqrt{2}$	N			
0.65		N		
1.27×10^4			Y	

- 6 Evaluate $\sqrt{\frac{32.76}{3.95 \times 2.63}}$, giving your answer:
- correct to 3 decimal places
 - correct to the nearest whole number
 - correct to 3 significant figures
 - in standard form.
- 7 The speed of light is approximately 186 280 miles per second. The distance from Mars to Earth is approximately 195 000 000 km.
- Given that mile : kilometre = 1 : 1.609, determine the speed of light in kilometres per minute. Give your answer to 3 significant figures.
 - Express your answer to **a** using scientific notation.
 - If a light source on Mars is ignited, how many minutes will it be before it can be seen through a telescope on Earth?
- 8 Consider the arithmetic sequence 120, a , 98, b ,
- Determine the values of a and b .
 - Write down the n th term of the sequence.
 - Determine the number of positive terms in this sequence.
- 9 Lisa invests £18 000 into a savings account which pays 4.5% p.a. interest, compounded annually.
- Calculate the value of Lisa's investment after:
 - 1 year
 - 2 years.
 - Write down a formula which calculates the value of the investment after n years.
 - Calculate the value of the investment after 13 years.
 - Determine the minimum length of time required for the investment to be triple its original value.
- 10
- A measurement of 5.645 cm is rounded to 3 significant figures.
 - Write down the actual error caused by rounding.
 - Calculate the percentage error.
 - The speedometer of a car reads 70 km h^{-1} . It is accurate to within 3.2%.
 - What is the maximum possible error?
 - Write down the extreme possible values for the true speed of the car.
- 11 The first three terms of a sequence are -2 , -9 , and -16 .
- Write down the next two terms of the sequence.
 - Draw a mapping diagram of the first 5 terms.

c Find a formula for the n th term of the sequence.

- 12
- Write down the first 3 terms of the sequence given by $u_n = n(n + 1)$.
 - Find the 15th term.
 - Which term of this sequence is 600?
- 13 Complete the following table by placing ticks in the appropriate boxes.

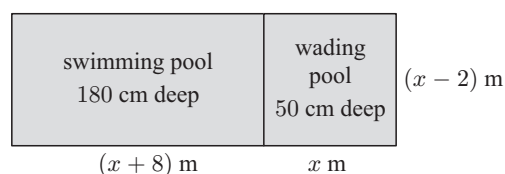
	$\sqrt{4}$	-2	3.75	π	$2.\overline{33}$
N					
Z					
Q					
R					

- 14 Find the value of $2.34 + \frac{5.25}{3.10 \times 7.65}$, giving your answer:
- correct to the nearest integer
 - correct to 4 decimal places
 - correct to 3 significant figures
 - in scientific notation.
- 15 The first three terms of an arithmetic sequence are -347 , $k - 166$ and -185 .
- Find the value of k .
 - Find a formula for the n th term of the sequence.
 - Which is the first positive term of the sequence?
- 16 The sixth term of an arithmetic sequence is 49 and the fifteenth term is 130.
- Find the common difference for this sequence.
 - Find the first term.
 - How many terms of this sequence have a value less than 300?
- 17 A section of tubing is formed into a circle with diameter 4.5 m.
- Calculate the length of the tubing.
 - Round your answer to the nearest whole number.
 - Calculate the percentage error if the length is rounded to the nearest whole number.
- 18 The first three terms of a geometric sequence are 0.75, 2.25 and 6.75.
- Find the common ratio.
 - Write down a formula for the n th term.
 - Calculate the sum of the first 10 terms.
- 19 The sum of the first 7 terms of an arithmetic series is 329. The common difference is 14.
- Find the value of the first term.
 - Find n given 69 800 is the sum of the first n terms of the sequence.
- 20 One stage of a cycling race covers 80 km on the flat, and the remainder in the mountains. The cyclists can average 45 km h^{-1} on the flat and 25 km h^{-1} in the mountains.
- Determine the time taken by the cyclists to cover the flat section of the course. Give your answer correct to the nearest minute.
 - If the total time taken for the stage is 3 hours, find the distance covered in the mountains.

SKILL BUILDER - LONG QUESTIONS

- 1 a** The n th term of an arithmetic sequence is given by $u_n = 4 + 11n$. Find:
- the first 2 terms of the sequence
 - the sum of the first 10 terms.
- b** The n th term of a geometric sequence is given by $u_n = 4(2.2)^{n-1}$. Find:
- the fifth term of the sequence
 - the sum of the first 10 terms.
- c** Find the first term for which the value of the geometric sequence is greater than the value of the arithmetic sequence. What is the difference between the terms of the two sequences for this value of n ?
- 2 a** Draw a number line to represent $x \in \mathbb{R}$, and mark the integers from -5 to 5 . Represent the following on the number line using appropriate notation.
- $x \geq 4$
 - $1 \leq x < 3$
 - $-3 \leq x \leq 0$
 - $x < -4$
- b** Let $X = \{x \mid -4 \leq x \leq 4, x \in \mathbb{Z}\}$ and $Y = \{y \mid -4 \leq y \leq 4, y \in \mathbb{Z}\}$.
On a set of coordinate axes, plot the points which represent the set:
- $W = \{(x, y) \mid \frac{y}{x} = -1, x \in X, y \in Y\}$
 - $V = \{(x, y) \mid x \geq 0, y \geq 0, y - x = 1, x \in X, y \in Y\}$
- c** If $x, y \in \mathbb{Q}$ and $x > y$, write down an example which makes $x^2 < y^2$ true.
- 3** The fifth term of an arithmetic sequence is 51, and the sum of the first 5 terms is 185.
- If u_1 is the first term of the sequence, and d is the common difference, write down two equations in u_1 and d which satisfy the information provided.
 - Solve the equations from **a** to determine the values of u_1 and d .
 - Find the first term in this sequence to exceed 1000.
 - The sum of the first k terms in this sequence is 3735.
 - Deduce that k satisfies the equation $7k^2 + 39k - 7470 = 0$.
 - Hence, find k .
- 4 a** Which of the following statements are false? Justify your answers.
- $\{-2, -1, 0, 1, 2\} \subset \{x \mid x < 2, x \in \mathbb{R}\}$
 - $\{0, 1, 2, 3, 4\} \subset \{x \mid x \leq 5, x \in \mathbb{Z}\}$
 - $\{x \mid x^2 + x = 2, x \in \mathbb{Z}\} = \{-1, 1, 2\}$
- b** Let $U = \{x \mid 14 \leq x < 30, x \in \mathbb{N}\}$,
 $A = \{\text{multiples of } 7\}$, $B = \{\text{factors of } 56\}$,
 $C = \{\text{even numbers } \geq 20\}$.
- List the members of each of the sets A , B and C contained in U .
 - Represent these sets on a 3-circle Venn diagram.
- c** p and q are different integers. Which of the following statements are false? Give an example to support your decision.
- $p + q = q + p$
 - $p - q = q - p$
 - $pq = qp$

- 5 a** Evaluate $25.32 \times \frac{6.057}{2.4 \times \sqrt{5.14}}$, giving your answer correct to:
- five significant figures
 - the nearest tenth
 - 1 significant figure.
- b** Three sections of fencing are erected. Each section has a stated length of 3.60 m, measured to the nearest tenth of a metre. The actual length of each section is 3.63 m.
- Find the actual length covered by the three sections of fencing.
 - Calculate the percentage error between the actual length and the stated length of the three sections of fencing.
- c** The three sections of fencing in **b** form one side of a square enclosure. The enclosure will have a concrete floor 100 ± 5 mm thick. Concrete costs €47.50 per cubic metre.
- Write down the minimum and maximum possible values for the volume of concrete needed for the floor.
 - Calculate the difference in cost between the maximum possible volume of concrete and the planned volume based on the stated length.
 - Express the cost difference in **ii** as a percentage of the planned cost.
- 6 a** Write down the first three terms of the following sequences:
- $u_n = 120 + 3(n - 1)$
 - $u_{n+1} = u_n + 7, u_1 = 4$.
- b** Which term do the sequences in **a** have in common, and what is the value of that term?
- c** Which of the sequences in **a** has 151 as one of its terms?
- d** The sum of the first n terms of the sequences in **a** is the same. Find n .
- e** When will the sum of the first sequence in **a** exceed the sum of the second sequence by 228?
- 7** A pool has two rectangular sections, the swimming pool that is 180 cm deep, and the wading pool that is 50 cm deep. The dimensions of these regions are shown in the diagram below.



- Show that the total area of the surface of the pool can be found by the expression $2x^2 + 4x - 16$.
 - Given that the total surface area of the pool is 224 m^2 , find the value of x .
 - Calculate the volume of water (in litres) that the pool contains.
 - Over a period of time, 2400 litres of water evaporates from the wading pool.
 - Find the volume of water in litres which the wading pool now contains.
 - Find the new depth of the wading pool.
- 8 a** The first 3 terms of a sequence are 56, 28 and 14.
- Show that the sequence is geometric.
 - Find the 8th term of the sequence.
 - Find the sum of the first 8 terms.

- b** The third term of another geometric sequence is 24.5 and the 5th term is 12.005. All of the terms of this sequence are positive.
- Find the first term and the common ratio.
 - Write down the general formula for a term of this sequence.
- c** The first n terms of the sequence in **b** are larger than the corresponding terms for the sequence $u_n = 20 \times (0.8)^{n-1}$. Find n .
- d** For the sequence $u_n = 20 \times (0.8)^{n-1}$, find the sum of the first:
- 30 terms
 - 50 terms
 - 100 terms.
- Answer to 3 significant figures in each case. Comment on your results.
- 9 a** Misha takes out a loan to purchase a generator for his business. He borrows \$20 000 at 12.5% per annum, compound interest. At the end of each year, Misha is required to pay \$ k .
- If Misha does repay \$ k each year, explain why the amount owing on the loan at the end of the first year is $\$(20\,000 \times 1.125 - k)$.
 - Write down an expression, in terms of k , for the amount owing on the loan at the end of the second year.
 - At the end of the second year, the amount owing on the loan will be \$17 131.25. Find k .
- b** Misha paid a total of \$24 000 for the generator. Its value depreciated after purchase so that, at the end of the first year, it was only worth \$20 400. The value after each subsequent year declines at the same rate.
- Find the percentage decrease in the value of the generator in the first year.
 - Calculate the value of the generator at the end of the second year.
 - Write down the formula for the value after n years.
 - Sketch the graph of the value of the generator for the first 6 years.
- 10 a** Consider the arithmetic sequence defined by $A_n = 19n - 13$. Write down:
- the first term
 - the common difference.
- b** G_n is a geometric sequence with 3rd term 1920 and 10th term 15.
- Calculate the common ratio for the sequence G_n .
 - Show that $G_1 = 7680$.
- c** By considering the equation $A_n = G_n$, find the term which A_n and G_n have in common.
- d** Let S_{A_n} represent the sum of the first n terms of A_n , and S_{G_n} represent the sum of the first n terms of G_n .
- Show that $S_{A_n} = \frac{19}{2}n^2 - \frac{7}{2}n$.
 - Write down an expression for S_{G_n} in terms of n .
 - Determine the minimum number of terms required for the sum of the first n terms of A_n to exceed the sum of the first n terms of G_n .
- 11** Tahlia borrows \$5000 from her bank. Her repayments each month are \$250 plus the interest for that month. Interest is charged at 1.2% each month.
- a** Determine the payments she must make at the end of the:
- first month
 - second month
 - third month.
- b** The monthly payments form an arithmetic sequence. Find the first term and common difference for the sequence. Hence write down the formula to find the value of the n th payment for this loan.
- c** Calculate the size of the payment at the end of the tenth month.
- d** Determine the number of payments required for Tahlia to pay off the loan.
- e** Calculate the total amount Tahlia will pay for this loan.
- 12** A radioactive material loses 10% of its weight per year. The weight of the material at the start of the first year was 200 g.
- Write down the weight of the material at the start of the second and third years.
 - The weights at the start of each year form a geometric sequence. Write down the common ratio of the sequence.
 - Find the weight of the material at the start of the 6th year.
 - Sketch the graph of the annual weight of the material for the first 6 years.
 - At the start of which year will the weight of the material fall below 20 g?
 - The weight of a second sample of radioactive material decreased from 120 g to 49.152 g after 6 years. Find the annual percentage rate of decrease.

TOPIC 3: SETS, LOGIC AND PROBABILITY

SETS

A **set** is a collection of numbers or objects.

Every number or object in a set is called an **element** or **member** of the set.

The **empty set** $\{ \}$ or \emptyset contains no elements.

The number of elements in set A is $n(A)$.

A set which has a finite number of elements is called a **finite set**.

An **infinite set** has infinitely many elements.

The **universal set** U is the set of all elements under consideration.

P is a subset of Q if every element of P is also an element of Q .	$P \subseteq Q$
P is a proper subset of Q if P is a subset of Q but is not equal to Q .	$P \subset Q$
The intersection of P and Q consists of all elements which are in both P and Q .	$P \cap Q$
The union of P and Q consists of all elements which are in P or Q or both.	$P \cup Q$
P and Q are disjoint or mutually exclusive if they have no elements in common.	
The complement of P , denoted P' , is the set of all elements of U which are <i>not</i> in P .	$P \cap P' = \emptyset$ $P \cup P' = U$

VENN DIAGRAMS

A **Venn diagram** consists of a universal set U represented by a rectangle.

Sets within the universal set are usually represented by circles.