TOPIC 1: ALGEBRA

SEQUENCES AND SERIES

A **number sequence** is a set of numbers defined by a rule which is valid for positive integers. Often, the rule is a formula for the **general term** or **nth term** of the sequence.

Arithmetic Sequences

In an **arithmetic sequence**, each term differs from the previous one by the same fixed number.

$u_{n+1} - u_n = d$ for all n, where d is a constant called the **common difference.**

For an arithmetic sequence with first term u_1 and common difference d, $u_n = u_1 + (n-1)\,d$.

Geometric Sequences

In a **geometric sequence**, each term is obtained from the previous one by multiplying by the same non-zero constant.

$\dfrac{u_{n+1}}{u_n} = r$ for all n, where r is a constant called the **common ratio.**

For a geometric sequence with first term u_1 and common ratio r, $u_n = u_1 r^{n-1}$.

Series

A **series** is the addition of the terms of a sequence. Given a series which includes the first n terms of a sequence, its sum is $S_n = u_1 + u_2 + ... + u_n$.

For an **arithmetic series**, $S_n = \frac{n}{2}(u_1 + u_n)$.

For a **geometric series**, $S_n = \dfrac{u_1(r^n - 1)}{r-1}$.

The sum of an **infinite geometric series** is
$S = \dfrac{u_1}{1-r}$ provided $|r| < 1$.
If $|r| > 1$ the series is **divergent**.

For compound interest problems we have a geometric sequence. If the interest rate is $i\%$ per time period then the common ratio is $\left(1 + \frac{i}{100}\right)$ and the number of compounding periods is n.

EXPONENTIALS AND LOGARITHMS

Exponential and logarithmic functions are inverses of each other. The graph of $y = \log_a x$ is the reflection in the line $y = x$ of the graph of $y = a^x$.

Exponent Laws	Logarithm Laws
$a^x \times a^y = a^{x+y}$	$\log_a xy = \log_a x + \log_a y$
$\dfrac{a^x}{a^y} = a^{x-y}$	$\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$
$(a^x)^y = a^{xy}$	$\log_a x^y = y \log_a x$
$a^0 = 1 \ \ (a \neq 0)$	$\log_a 1 = 0$
$a^1 = a$	$\log_a a = 1$

If $a^x = a^k$ then $x = k$.

To change the base of a logarithm, use the rule $\log_b x = \dfrac{\log_c x}{\log_c b}$.

COMPLEX NUMBERS

Any number of the form $a + bi$ where a and b are real and $i = \sqrt{-1}$ is called a **complex number**.

If $z = a + bi$ where a and b are real then:

- a is the **real part** of z or $\mathcal{R}e(z)$
- b is the **imaginary part** of z or $\mathcal{I}m(z)$

Two complex numbers are equal if their real parts are equal and their imaginary parts are equal.
$a + bi = c + di \;\Leftrightarrow\; a = c$ and $b = d$.

The **complex conjugate** of $z = a + bi$ is $z^* = a - bi$.

Properties of complex conjugates:

- $(z^*)^* = z$
- $(z_1 + z_2)^* = z_1^* + z_2^*$ and $(z_1 - z_2)^* = z_1^* - z_2^*$
- $(z_1 z_2)^* = z_1^* \times z_2^*$ and $\left(\dfrac{z_1}{z_2}\right)^* = \dfrac{z_1^*}{z_2^*}$, $z_2 \neq 0$
- $(z^n)^* = (z^*)^n$.

On the complex plane, $a + bi$ is represented as $\begin{pmatrix} a \\ b \end{pmatrix}$.

The **modulus** of the complex number $z = a + bi$ is the real number $|z| = \sqrt{a^2 + b^2}$.

If z is represented by the point P(a, b) on the Cartesian plane, and $\overrightarrow{\text{OP}}$ makes angle θ with the positive real axis, then θ is called the **argument** of z, or $\arg z$.

Properties of modulus and argument:

- $|wz| = |w|\,|z|$ and $\arg(wz) = \arg w + \arg z$
- $\left|\dfrac{w}{z}\right| = \dfrac{|w|}{|z|}$ and $\arg\left(\dfrac{w}{z}\right) = \arg w - \arg z$
- $|z^*| = |z|$ and $\arg(z^*) = -\arg z$
- $zz^* = |z|^2$

Cartesian form $z = a + bi$

Polar form $z = |z| \operatorname{cis}\theta$ where $\operatorname{cis}\theta = e^{i\theta} = \cos\theta + i\sin\theta$

$\operatorname{cis} 0 = 1$, $\operatorname{cis}\frac{\pi}{2} = i$, $\operatorname{cis}\pi = -1$, $\operatorname{cis}\frac{3\pi}{2} = -i$

$\operatorname{cis}\theta \times \operatorname{cis}\phi = \operatorname{cis}(\theta + \phi)$

$\dfrac{\operatorname{cis}\theta}{\operatorname{cis}\phi} = \operatorname{cis}(\theta - \phi)$

$\operatorname{cis}(\theta + k2\pi) = \operatorname{cis}\theta$ for all $k \in \mathbb{Z}$.

De Moivre's Theorem

$\left(|z| \operatorname{cis}\theta\right)^n = |z|^n \operatorname{cis} n\theta$

The **nth roots of the complex number c** are the n solutions of $z^n = c$.

There are *exactly* n nth roots of c.

If $c \in \mathbb{R}$, the roots must occur in conjugate pairs, so if w is a complex, non-real zero of the real polynomial $P(z)$ then w^* is also a zero and $z^2 - (w + w^*)\,z + ww^*$ is a real quadratic factor of $P(z)$.

The nth roots of c will all have the same modulus, which is $|c|^{\frac{1}{n}}$.

The **nth roots of unity** are the n solutions of $z^n = 1$.

COUNTING AND THE BINOMIAL THEOREM

If there are m different ways of performing an operation and n different ways of performing a second independent operation, there are mn ways of performing the two operations in succession.

In counting processes, the word:

- **and** suggests multiplying the possibilities
- **or** suggests adding the possibilities.

Factorial notation

$n! = n(n-1)(n-2)\ldots \times 3 \times 2 \times 1$ for all $n \geqslant 1$.

$0! = 1$

Permutations and combinations

A **permutation** of a group of symbols is *any arrangement* of those symbols in a definite *order*.

A **combination** is a selection of objects *without regard to order* or arrangement.

The number of combinations on n distinct symbols taken r at a time is C_r^n or ${}_nC_r$. It may also be written as the **binomial coefficient** $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$.

- $\binom{n}{r} = \binom{n}{n-r}$
- $\binom{n}{0} = \binom{n}{n} = 1$
- $\binom{n}{1} = \binom{n}{n-1} = n$.

The **general binomial expansion** is

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n.$$

MATHEMATICAL INDUCTION

Suppose P_n is a proposition which is defined for every integer $n \geqslant a$, $a \in \mathbb{Z}$.

To construct a formal proof by mathematical induction:

- make sure the proposition is clearly stated
- prove the initial case P_a is true
- prove that if P_k is true then P_{k+1} must also be true
- state your conclusion clearly.

SKILL BUILDER QUESTIONS

1 Find the sum of the first 30 terms of:

a $18 + 16 + 14 + 12 + \ldots\ldots$
b $48 + 24 + 12 + 6 + \ldots.$

2 Find the sum of the infinite geometric series with first term 27 and fourth term 8.

3 A sequence is defined by $u_n = \dfrac{2n+1}{3}$ for $n \in \mathbb{Z}^+$.

a Prove that the sequence is arithmetic.
b Find the 50th term and the sum of the first 50 terms.
c Is 117 a term of the sequence?
d Find:
i $\sum_{n=1}^{40} u_n$ **ii** $\sum_{n=30}^{60} u_n$.

4 The 7th and 15th terms of an arithmetic sequence are 1 and -23 respectively. Find:

a the 27th term
b the sum of the first 27 terms of the sequence.

5 The sum of an infinite geometric series is 1.5, and its first term is 1. Find:

a the common ratio
b the sum of its first 7 terms in rational form.

6 A sequence has consecutive terms $k+1$, $3k$ and k^2+5 where $0 < k < 5$.
Find k if the sequence is:

a arithmetic **b** geometric.

7 A sequence is defined by $u_n = 12\left(\frac{2}{3}\right)^{n-1}$.

a Prove that the sequence is geometric.
b Find the 5th term in rational form.
c Find:
i $\sum_{n=1}^{\infty} u_n$
ii $\sum_{n=1}^{20} u_n$ correct to 4 decimal places.

8 u_n is a geometric sequence in which $u_3 = 20$ and $u_6 = 160$. Find:

a u_1 and the common ratio
b u_{10} and $\sum_{n=1}^{12} u_n$.

9 Three numbers are consecutive terms of an arithmetic sequence. Find the numbers given that their sum is 18 and the sum of their squares is 396.

10 The first term of a finite arithmetic series is 18 and the sum of the series is -210. If the common difference is -3, find the number of terms in the series.

11 Stan invests £3500 for 33 months at an interest rate of 8% p.a. compounded quarterly. What will be its maturing value?

12 Find the sum of all integers between 100 and 200 (inclusive) which are **not** divisible by 4.

13 Find:

a n if $\sum_{k=1}^{n}(3k-11) = 5536$
b y if $\sum_{k=1}^{\infty}\left(\dfrac{y}{5}\right)^{k-1} = 5$.

14 Write $\dfrac{x^a\sqrt{x^{3a}}}{x^{-2a}}$ as a single power of x.

15 Write in simplest form: $\left(\dfrac{3x^{-1}}{2a^2}\right)^{-2} \times \left(\dfrac{4x^2}{27a^{-3}}\right)^{-1}$

16 Find x if $8^{2x-3} = 16^{2-x}$.

17 Simplify: $\dfrac{3^{x+1}-3^x}{2(3^x)-3^{x-1}}$

18 Solve for x: $4^x + 4 = 17(2^{x-1})$

19 Find a and b given that $2^a 8^b = \frac{1}{2}$ and $\dfrac{3^{-a}}{3^{b+1}} = 9$.

20 Find: **a** $\log_4 8$ **b** $\log_9\left(\frac{1}{27}\right)$ **c** $\log_{\frac{1}{3}}(\sqrt{3})$

21 If $\log_5(2x-1)=-1$, find x.

22 Write $\dfrac{8}{\log_5 9}$ in the form $a\log_3 b$ where $a, b \in \mathbb{Z}$.

23 Write $2\ln x + \ln(x-1) - \ln(x-2)$ as a single logarithm.

24 Solve for x: $\log_3 x + \log_3(x-2) = 1$.

25 If $\log_a 5 = x$, find in terms of x:

a $\log_a(5a)$ **b** $\log_a\left(\dfrac{a^2}{25}\right)$.

26 Write as a logarithmic equation in base b:

a $M = ab^3$ **b** $D = \dfrac{a}{b^2}$

27 Write without logarithms:

a $\log_{10} M = 2x - 1$

b $\log_a N = 2\log_a d - \log_a c$

28 Solve for x: $x^2 > e^{-x}$.

29 The solution of $2^{x-1} = 3^{2-x}$ is $x = \log_a b$ where $a, b \in \mathbb{Z}^+$. Find a and b.

30 Expand and simplify: $(2-ai)^3$

31 Write $3 - 3i\sqrt{3}$ in the form $r(\cos\theta + i\sin\theta)$.

32 Simplify $\left(\cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)\right)^{10}$, giving your answer in the form $x+iy$ where $x, y \in \mathbb{R}$.

33 Write in the form $a+bi$ where $a, b \in \mathbb{Q}$:

a $\dfrac{3+4i}{1-3i}$ **b** $\dfrac{3}{i}\left(\frac{1}{\sqrt{5}} - \frac{2i}{\sqrt{5}}\right)^2$

34 If $\dfrac{z+2}{z-2} = i$, find z in the form $a+bi$ where $a, b \in \mathbb{R}$.

35 z is a complex number and z^* is its complex conjugate. Show that if $z^2 = (z^*)^2$ then z is either real or purely imaginary.

36 **a** Find the roots of $z^5 = 1$ and display them on a fully labelled Argand diagram.

b If the roots found in **a** are $1, w, w^2, w^3$ and w^4 where w is the root with smallest positive argument, show that $1 + w + w^2 + w^3 + w^4 = 0$.

37 Solve for z: $z^2 - z + 1 + i = 0$

38 Prove that $(z+w)^* = z^* + w^*$.

39 Prove that $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$.

40 For what values of k does $\dfrac{z^2+3}{z^2-1} = k$ have imaginary roots?

41 Illustrate $\{z \mid \mathcal{Re}(z) \leqslant 1 \cap 0 \leqslant \mathcal{Im}(z) \leqslant 3\}$.

42 Write $1-i$ in polar form, and hence find $(1-i)^{11}$ in Cartesian form.

43 Write $z = \dfrac{-1+5i}{2+3i}$ in polar form, and hence show that $z^{12} = -64$.

44 If z and w are complex numbers where $w \neq 0$, use polar coordinates to show that $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$.

45 If $z = x+iy$ where $x, y \in \mathbb{R}$ and $|z-3| = |z-1|$, deduce that $x = 2$.

46 Suppose $z_1 = 3+4i$ and $z_2 = 8-8i$.

a Find $|z_1|$, $\arg z_1$, $|z_2|$ and $\arg z_2$.

b Show the points $P(z_1)$, $Q(z_2)$ and $R(z_1+z_2)$ on the complex plane and explain how point R is located geometrically from P and Q.

47 Suppose $z = r\,\text{cis}\,\theta$ where $r > 0$, and $z^2 = z^*$.

a Deduce that $r^2 = r$ and $\text{cis}\,3\theta = 1$.

b Hence show that $z^2 = z^*$ has three non-zero solutions and write them in the form $a+bi$ where $a, b \in \mathbb{R}$.

48 If $z = iz^*$ where $z = x+iy$ and $x, y \in \mathbb{R}$, deduce that $x = y$.

49 Given that $\text{cis}\,\theta = e^{i\theta}$, deduce that:

a $\text{cis}\,\theta\,\text{cis}\,\phi = \text{cis}\,(\theta+\phi)$ **b** $(r\,\text{cis}\,\theta)^n = r^n\text{cis}\,(n\theta)$

c if $w = e^{i\left(\frac{2\pi}{5}\right)}$, then $(1+w)(1+w^2) = -w^4$.

50 **a** Write $z = \dfrac{1+i\sqrt{3}}{1+i}$ in the form $r\,\text{cis}\,\theta$, $r > 0$.

b Hence, find the smallest positive value of n such that

i $z^n \in \mathbb{R}$ **ii** z^n is purely imaginary.

51 Suppose $w = e^{i\left(\frac{2\pi}{3}\right)}$.

a Deduce that $w^3 = 1$ and $1 + w + w^2 = 0$.

b Write in terms of w in simplest form:

i w^7 **ii** w^{-1} **iii** $(1-w)^2$

iv $\dfrac{1}{(1+w)^2}$ **v** $\dfrac{1+w^2}{1+w}$

52 **a** Find the cube roots of $-27i$ and display them on an Argand diagram, labelling them z_1, z_2 and z_3.

b Show that $z_2z_3 = z_1^2$, where z_1 is any of the cube roots found in **a**.

c What is the value of $z_1z_2z_3$?

53 Use complex number methods to write $\sin 3\theta$ in the form $a\sin\theta + b\sin^3\theta$.

54 **a** If $z = \text{cis}\,\theta$, prove that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ for $n \in \mathbb{Z}^+$

b By considering $\left(z + \dfrac{1}{z}\right)^4$, write $\cos^4\theta$ in the form $a\cos 4\theta + b\cos 2\theta + c$ where a, b and $c \in \mathbb{Q}$.

55 Consider the binomial expansion of $\left(2x - \dfrac{1}{x^2}\right)^{12}$. Find:

a the coefficient of x^3 **b** the constant term.

56 Prove that $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$ for positive integers n and r, $r \leqslant n$.

57 In a party game each person has a card with the 12 numbers from 1 to 12 printed on it. Each person is to put a cross through 3 of the numbers, for example 2, 7 and 10.
A prize is won if the three numbers crossed are the same as the three numbers chosen by the host.
How many different possible combinations of three numbers are there?

58 In how many integers between 1000 and 9999 inclusive does the digit 5 occur?

59 The coefficient of x^3 in the expansion of $(3x+2)^n$ is 21 times the coefficient of x. Find n.

60 Consider all 4-digit integers where all the digits are different and the first digit is non-zero.

a How many of these numbers are there?

b How many of these numbers have a "7" as one of the four digits?

61 Find k given that the constant term of

$$\left(kx+\frac{1}{\sqrt{x}}\right)^9 \text{ is } -10\tfrac{1}{2}.$$

62 At a reunion between 6 men and 5 women, each person shakes hands once with every other person. Find:

a the total number of handshakes

b the number of hand shakes between a man and a woman.

63 48 people are to travel in a double-decker bus which seats 24 on each level. However, 8 people refuse to travel upstairs and 6 refuse to travel downstairs. How many ways are there of choosing which passengers travel upstairs and which passengers travel downstairs?

64 Find the coefficient of x^5 in the expansion of $(x+2)(1-x)^{10}$.

65 Of 11 given points, 4 lie on a straight line and no other three points are collinear. How many different straight lines can be drawn through pairs of given points?

66 Prove that $\sum_{r=0}^{n}\binom{n}{r}=2^n$ for $n\in\mathbb{Z}^+$

67 A teacher needs to decide the order in which to schedule 8 examinations, two of which are Mathematics A and Mathematics B. In how many ways can this be done given that the two Mathematics subjects must not be consecutive?

68 By considering $(1+x)^{2n}=(1+x)^n(1+x)^n$ show that for $n\in\mathbb{Z}^+$,

$$\binom{2n}{n}=\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+....+\binom{n}{n}^2.$$

69 Deduce that $n\binom{n-1}{r-1}=r\binom{n}{r}$ for $n,\ r\in\mathbb{Z}^+,\ n\geqslant r$.

70 A club has 12 members. How many different committees consisting of at least two members can be formed?

71 Find b given that the coefficient of x^{-3} in

$$\left(\sqrt{x}+\frac{b}{x}\right)^9 \text{ is } -4032.$$

72 5 distinct points lie on a circle and 11 distinct points lie within it. No three points are collinear. How many different triangles can be drawn with vertices selected from the 16 points if:

a there are no other restrictions

b exactly one of the vertices lies on the circle

c at least one of the vertices lie within the circle.

73 Prove that $1\times2+2\times5+3\times8+......+n(3n-1)=n^2(n+1)$ for $n\in\mathbb{Z}^+$.

74 Prove that $5n^3-3n^2-2n$ is divisible by 6 for $n\in\mathbb{Z}^+$.

75 **a** There is an integer b such that $0<b\leqslant 9$ and 9^n+b is divisible by 8 for $n\in\mathbb{Z}^+$. Find the value of b.

b Prove your answer in **a** using the principle of mathematical induction.

76 Prove that $1+2\times2^1+3\times2^2+......+n\times2^{n-1}=(n-1)2^n+1$ for $n\in\mathbb{Z}^+$.

77 **a** Show that $4x^2\geqslant(x+1)^2$ for $x\geqslant1$.

b Use the principle of mathematical induction to prove that $4^n\geqslant 3n^2$ for $n\in\mathbb{Z}^+$.

78 Deduce that n^3+2n is divisible by 3 for $n\in\mathbb{Z}^+$.

79 Use the principle of mathematical induction to prove that $3^n>n^2+n$ for $n\in\mathbb{Z}^+$.

TOPIC 2: FUNCTIONS AND EQUATIONS

FUNCTIONS $f:x\mapsto f(x)$ OR $y=f(x)$

A **relation** is any set of points on the Cartesian plane.

A **function** is a relation in which no two different ordered pairs have the same x-coordinate or first member. So, for each value of x there is only one value of y or $f(x)$. We sometimes refer to y or $f(x)$ as the **image value** of x.

We **test for functions** using the vertical line test. A graph is a function if no vertical line intersects the graph more than once. For example, a circle such as $x^2+y^2=1$ has a graph which is not a function.

The **domain** of a function is the set of values that x can take.

To find the domain of a function, remember that we cannot:

- divide by zero
- take the square root of a negative number
- take the logarithm of a non-positive number.

The **range** of a function is the set of values that y or $f(x)$ can take.

A relation is:

- **one-to-one** if there is only one y for each x *and* only one x for each y.
- **many-to-one** if there there is more than one value of x with the same value of y.

Given $f:x\mapsto f(x)$ and $g:x\mapsto g(x)$, the **composite function** of f and g is $f\circ g:x\mapsto f(g(x))$.

In general, $f(g(x))\neq g(f(x))$, so $f\circ g\neq g\circ f$.

The **absolute value** or **modulus** function

$$f(x)=|x|=\begin{cases} x & \text{if } x\geqslant 0\\ -x & \text{if } x<0.\end{cases}$$

Properties of modulus for all x and y:

- $|x|\geqslant 0$
- $|-x|=|x|$
- $|x|^2=x^2$
- $|xy|=|x|\,|y|$
- $\left|\frac{x}{y}\right|=\frac{|x|}{|y|}$
- $|x-y|=|y-x|$

The **identity function** is $f(x)=x$.

The function $y=f(x)$ has an **inverse function** $y=f^{-1}(x)$ if and only if it is one-to-one.
A many-to-one function will not have an inverse.