b Let $u = 3x - 4$

$\therefore \quad x = \dfrac{u+4}{3}$ and $\dfrac{du}{dx} = 3.$

$$\therefore \quad \int x\sqrt{3x-4}\,dx = \tfrac{1}{3}\int \frac{u+4}{3}\sqrt{u}(3)\,dx$$
$$= \tfrac{1}{9}\int \left((u+4)\sqrt{u}\right)du$$
$$= \tfrac{1}{9}\int (u^{\frac{3}{2}} + 4u^{\frac{1}{2}})\,du$$
$$= \tfrac{1}{9}\left(\tfrac{2}{5}u^{\frac{5}{2}} + \tfrac{8}{3}u^{\frac{3}{2}}\right) + c$$
$$= \tfrac{2}{45}(3x-4)^{\frac{5}{2}} + \tfrac{8}{27}(3x-4)^{\frac{3}{2}} + c$$

39 **a** $$\frac{x+2}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$$
$$= \frac{A(x+3)+B(x-1)}{(x-1)(x+3)}$$
$$= \frac{(A+B)x + 3A - B}{(x-1)(x+3)}$$

$\therefore \quad A + B = 1$ ...... (1)

$3A - B = 2$ ...... (2)

From (1), $B = 1 - A$

Substituting into (2), $3A - (1-A) = 2$

$\therefore \quad 4A = 3$

$\therefore \quad A = \frac{3}{4}$

$\therefore \quad B = \frac{1}{4}$

$$\therefore \quad \frac{x+2}{(x-1)(x+3)} = \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{4}}{x+3}$$

b $$\int \frac{x+2}{(x-1)(x+3)}\,dx = \int\left(\frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{4}}{x+3}\right)dx$$
$$= \tfrac{3}{4}\ln|x-1| + \tfrac{1}{4}\ln|x+3| + c$$

SOLUTIONS TO EXAMINATION PRACTICE SET 1

1 **a** $S_8 = -4$ and $S_{16} = 188 - 4 = 184$

But $S_n = \dfrac{n}{2}[2u_1 + (n-1)d]$

$\therefore \quad \frac{8}{2}[2u_1 + 7d] = -4$

$\therefore \quad 2u_1 + 7d = -1$ ...... (1)

and $\frac{16}{2}[2u_1 + 15d] = 184$

$\therefore \quad 8[2u_1 + 15d] = 184$

$\therefore \quad 2u_1 + 15d = 23$ ...... (2)

(2) $-$ (1) gives $8d = 23 - -1 = 24 \quad \therefore \quad d = 3$

and in (1), $2u_1 + 21 = -1 \quad \therefore \quad u_1 = -11$

Now $u_n = u_1 + (n-1)d$

$\therefore \quad u_n = -11 + (n-1)3$

$\therefore \quad u_n = 3n - 14$

b 9, x, y are geometric $\Rightarrow \quad \dfrac{x}{9} = \dfrac{y}{x}$ ...... (1)

x, y, 2 are arithmetic $\Rightarrow \quad y - x = 2 - y$ .... (2)

From (2) $2y = x + 2 \quad \therefore \quad y = \dfrac{x+2}{2}$

So, in (1), $\dfrac{x}{9} = \dfrac{x+2}{2x}$

$\therefore \quad 2x^2 = 9x + 18$

$\therefore \quad 2x^2 - 9x - 18 = 0$

$\therefore \quad (2x+3)(x-6) = 0$

$\therefore \quad x = -\frac{3}{2}$ or 6

$\therefore \quad x = 6 \qquad$ {since $x \in \mathbb{Z}$}

and so $y = \dfrac{6+2}{2} = 4$

2 $\overrightarrow{\text{OX}}$ is perpendicular to $\overrightarrow{\text{OY}}$ if $\overrightarrow{\text{OX}} \bullet \overrightarrow{\text{OY}} = 0$

$$\therefore \quad \begin{pmatrix} 3\mu \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} \mu \\ 1-3\mu \\ 2\mu-1 \end{pmatrix} = 0$$

$\therefore \quad 3\mu^2 + 2(1-3\mu) + (2\mu - 1) = 0$

$\therefore \quad 3\mu^2 - 4\mu + 1 = 0$

$\therefore \quad (3\mu - 1)(\mu - 1) = 0$

$\therefore \quad$ if $\mu = \frac{1}{3}$ or $\mu = 1$

3 **a** f is $y = x^2 + 4x \quad -\infty < x \leqslant -2$

so f^{-1} is $x = y^2 + 4y, \quad -\infty < y \leqslant -2$

$\therefore \quad y^2 + 4y - x = 0$

$$\therefore \quad y = \frac{-4 \pm \sqrt{16+4x}}{2} = -2 \pm \sqrt{4+x}$$

But $y \leqslant -2$, so $y = -2 - \sqrt{4+x}$

b $(g \circ f)(-3) = g\,(f(-3))$
$$= g\left((-3)^2 + 4(-3)\right)$$
$$= g(-3)$$
$$= \sqrt{3 - 2(-3)}$$
$$= \sqrt{3+6} = 3$$

4 Let $P(x) = x^n + ax^2 - 6$. From the remainder theorem

$P(1) = -3$ so $1 + a - 6 = -3$

$\therefore \quad a = 2$

Also $P(-3) = (-3)^n + a(-3)^2 - 6$

so $P(-3) = (-3)^n + 2 \times 9 - 6$

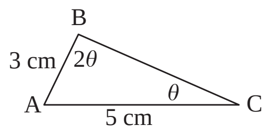
$= (-3)^n + 12$

$\therefore \quad (-3)^n + 12 = -15$ {given}

$\therefore \quad (-3)^n = -27$ and so $n = 3$

$\therefore \quad P(x) = x^3 + 2x^2 - 6$

5



By the sine rule, $\dfrac{\sin 2\theta}{5} = \dfrac{\sin\theta}{3}$

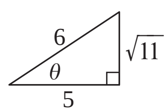
$\therefore \quad 3\sin 2\theta = 5\sin\theta$

$\therefore \quad 6\sin\theta\cos\theta = 5\sin\theta$

$\therefore \quad \sin\theta(6\cos\theta - 5) = 0$

$\therefore \quad$ as $\sin\theta \neq 0, \quad \cos\theta = \frac{5}{6}$

As $3\theta < 180^o$, $\theta < 60^o$. So, θ is acute.



$\therefore \quad \sin\theta = \frac{\sqrt{11}}{6}$

Let $\text{B}\widehat{\text{A}}\text{C} = \alpha^o$ then $\alpha + 3\theta = \pi$

$\therefore \quad \alpha = \pi - 3\theta$

$$\sin\alpha = \sin(\pi - 3\theta)$$
$$= \sin\pi\cos 3\theta - \cos\pi\sin 3\theta$$
$$= (0)\cos 3\theta - (-1)\sin 3\theta$$
$$= \sin 3\theta$$
$$= \sin(2\theta + \theta)$$
$$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$$
$$= 2\sin\theta\cos^2\theta + (\cos^2\theta - \sin^2\theta)\sin\theta$$
$$= 2\left(\tfrac{\sqrt{11}}{6}\right)\left(\tfrac{25}{36}\right) + \left(\tfrac{25}{36} - \tfrac{11}{36}\right)\left(\tfrac{\sqrt{11}}{6}\right)$$
$$= \tfrac{8}{27}\sqrt{11}$$

So, area $\Delta ABC = \frac{1}{2} \times 3 \times 5 \times \sin\alpha$

$= \frac{15}{2}\left(\frac{8}{27}\sqrt{11}\right)\ \text{cm}^2$

$= \left(\frac{60}{27}\sqrt{11}\right)\ \text{cm}^2$

6 $\sin^2 x + \cos x = \frac{5}{4}, \quad -\pi \leqslant x \leqslant \pi$

$\therefore\ 1 - \cos^2 x + \cos x - \frac{5}{4} = 0$

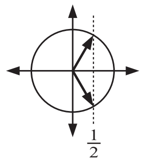
$\therefore\ \cos^2 x - \cos x + \frac{1}{4} = 0$

$\therefore\ 4\cos^2 x - 4\cos x + 1 = 0$

$\therefore\ (2\cos x - 1)^2 = 0$

$\therefore\ \cos x = \frac{1}{2}$

$\therefore\ x = \pm\frac{\pi}{3}$



7 $2^{2x} + 2^{x+1} - 15 = 0$

Let $m = 2^x \quad \therefore\ m^2 = (2^x)^2 = 2^{2x}$

So, $m^2 + 2m - 15 = 0$

$\therefore\ (m-3)(m+5) = 0$

$\therefore\ m = 3$ or -5

$\therefore\ 2^x = 3$ or -5

$\therefore\ 2^x = 3 \qquad \{2^x > 0 \text{ for all } x\}$

$\therefore\ \log 2^x = \log 3$

$\therefore\ x\log 2 = \log 3$

$\therefore\ x = \dfrac{\log 3}{\log 2} \quad (\text{or } \log_2 3)$

8 **a** $\dfrac{p(x)}{x(2x-3)} = (ax+b) + \dfrac{ax+b}{x(2x-3)}$

$= \dfrac{(ax+b)x(2x-3) + (ax+b)}{x(2x-3)}$

$\therefore\ p(x) = (ax+b)x(2x-3) + (ax+b)$

$\therefore\ p(x) = (ax+b)\left[x(2x-3)+1\right]$

$\therefore\ p(x) = (ax+b)(2x^2 - 3x + 1)$

$\therefore\ p(x) = (ax+b)(2x-1)(x-1)$

b Since $p\left(\frac{1}{2}\right) = 0$ and $p(1) = 0$, both $(2x-1)$ and $(x-1)$ are factors of $p(x)$.

c $p(0) = 7 \quad \Rightarrow \quad b(-1)(-1) = 7$

$\therefore\ b = 7$

$p(2) = 39 \quad \Rightarrow \quad (2a+7)(4-1)(2-1) = 39$

$\therefore\ (2a+7)3 = 39$

$\therefore\ 2a + 7 = 13$

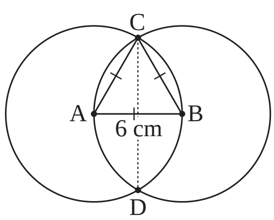
$\therefore\ 2a = 6$

$\therefore\ a = 3$

$\therefore\ p(x) = (3x+7)(2x-1)(x-1)$

$= 6x^3 + 5x^2 - 18x + 7$

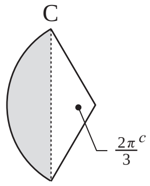
9



ΔABC and ABD are equilateral

$\therefore\ \widehat{CBD} = 120^o = \frac{2\pi}{3}^c$

Thus, total area $= 2 \times$ shaded area



Area of segment $= \frac{1}{2}(r^2\theta - r^2\sin\theta)$

$\therefore$ required area $= r^2\theta - r^2\sin\theta$

$= 6^2\left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right)\right)$

$= 36\left(\frac{2\pi}{3} - \left(\frac{\sqrt{3}}{2}\right)\right)$

$= 24\pi - 18\sqrt{3}\ \text{cm}^2$

10 By the cosine rule, $7^2 = x^2 + 5^2 - 2 \times x \times 5 \times \cos 60^o$

$\therefore\ 49 = x^2 + 25 - 10x\left(\frac{1}{2}\right)$

$\therefore\ x^2 - 5x - 24 = 0$

$\therefore\ (x-8)(x+3) = 0$

$\therefore\ x = 8$ or -3

But $x > 0$, so $x = 8$

11 **a** Since this is a probability distribution,

then $\sum P(X = x) = 1$

$\therefore\ \frac{1}{6} + 2k + \frac{1}{5}k + \frac{1}{3} + \frac{2}{5}k = 1$

$\therefore\ \frac{1}{6} + \frac{1}{3} + \left(2 + \frac{1}{5} + \frac{2}{5}\right)k = 1$

$\therefore\ \frac{1}{2} + \frac{13}{5}k = 1$

$\therefore\ k = \frac{5}{26}$

b $P(0 < X < 4) = P(X = 1, 2, 3)$

$= 2k + \frac{1}{5}k + \frac{1}{3}$

$= \left(2 + \frac{1}{5}\right)\frac{5}{26} + \frac{1}{3}$

$= (\frac{11}{5})(\frac{5}{26}) + \frac{1}{3}$

$= \frac{59}{78}$

12 $y = \sin x$ has $\dfrac{dy}{dx} = \cos x$

When $x = \frac{\pi}{6}$, $\dfrac{dy}{dx} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

$\therefore$ normal has slope $\frac{-2}{\sqrt{3}}$ and so the equation of the normal is $y - \frac{1}{2} = \frac{-2}{\sqrt{3}}\left(x - \frac{\pi}{6}\right)$.

The normal meets the x-axis when $y = 0$

$\therefore\ -\frac{1}{2} = \frac{-2}{\sqrt{3}}\left(x - \frac{\pi}{6}\right)$

$\therefore\ x - \frac{\pi}{6} = \frac{\sqrt{3}}{4}$

$\therefore\ x = \frac{\sqrt{3}}{4} + \frac{\pi}{6} \quad \therefore$ coordinates are $(\frac{\sqrt{3}}{4} + \frac{\pi}{6}, 0)$.

13 $f(x) = ax^3 + bx^2 + cx + d$

$\therefore\ f'(x) = 3ax^2 + 2bx + c$

Now $f(0) = 1, \qquad f'(0) = 0$

$f(-2) = -2, \qquad f'(-2) = 0$

Using $f(0) = 1$ we have $d = 1$.

Using $f'(0) = 0$ we have $3a(0) + 2b(0) + c = 0$.

So, $c = 0$, $d = 1$

$f(-2) = -2 \quad \therefore\ a(-8) + b(4) + 0(-2) + 1 = -2$

$\therefore\ -8a + 4b + 1 = -2$

$\therefore\ -8a + 4ba = -3 \quad \text{.... (1)}$

$f'(-2) = 0 \quad \therefore\ 3a(4) + 2b(-2) + 0 = 0$

$\therefore\ 12a - 4b = 0$

$\therefore\ b = 3a \quad \text{.... (2)}$

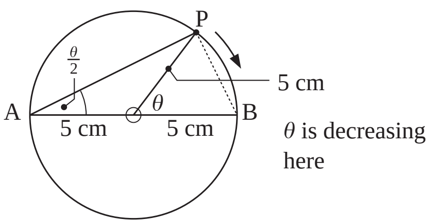
Substituting (2) into (1), $-8a + 12a = -3$

$\therefore\ 4a = -3$

$\therefore\ a = -\frac{3}{4}$ and so $b = -\frac{9}{4}$

Thus $a = -\frac{3}{4}$, $b = -\frac{9}{4}$, $c = 0$, $d = 1$.

14



$$\frac{d\theta}{dt} = \frac{-1 \text{ rev}}{10\pi \text{ s}} = \frac{-2\pi^c}{10\pi \text{ s}} = -\tfrac{1}{5}{}^c\,\text{s}^{-1}$$

$\widehat{\text{PAB}} = \frac{\theta}{2}$ {angle at centre theorem}

$A = \frac{1}{2}(\text{AP})(\text{PB})$

$\therefore \quad A = \frac{1}{2}\left[10\cos\left(\frac{\theta}{2}\right)\right]\left[10\sin\left(\frac{\theta}{2}\right)\right]$

$\therefore \quad A = 50\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$

$\therefore \quad A = 25\sin\theta$

$\therefore \quad \dfrac{dA}{dt} = 25\cos\theta\,\dfrac{d\theta}{dt}$

Particular case:

When $\frac{\theta}{2} = \frac{\pi}{3}, \quad \theta = \frac{2\pi}{3}$

$\therefore \quad \dfrac{dA}{dt} = 25\cos\left(\frac{2\pi}{3}\right)\left(-\frac{1}{5}\right) = -5\left(-\frac{1}{2}\right) = 2.5$

$\therefore$ at this instant the area is increasing at 2.5 cm^2 per second.

15 $\displaystyle\int_0^{\frac{\pi}{6}} (\sin 3x\cos 2x + \cos 3x\sin 2x)\,dx$

$= \displaystyle\int_0^{\frac{\pi}{6}} \sin(5x)\,dx$

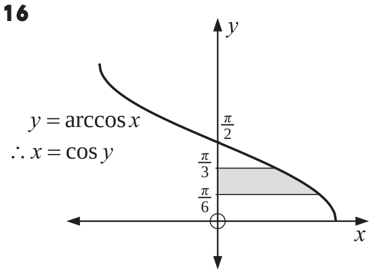
$= \frac{1}{5}\left[-\cos(5x)\right]_0^{\frac{\pi}{6}}$

$= -\frac{1}{5}\left(\cos\left(\frac{5\pi}{6}\right) - \cos 0\right)$

$= -\frac{1}{5}\left(\frac{-\sqrt{3}}{2} - 1\right)$

$= \frac{\sqrt{3}}{10} + \frac{1}{5}$

16



$V = \pi \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} x^2\,dy$

$= \pi \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 y\,dy$

$= \pi \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2} + \frac{1}{2}\cos 2y\,dy$

$= \pi\left[\frac{1}{2}y + \frac{1}{4}\sin 2y\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \pi\left(\frac{\pi}{6} + \frac{1}{4}\sin\left(\frac{2\pi}{3}\right) - \frac{\pi}{12} - \frac{1}{4}\sin\left(\frac{\pi}{3}\right)\right)$

$= \pi\left(\frac{\pi}{12}\right)$

$= \frac{\pi^2}{12}$ units^3

17 Let $x = \sin u, \quad \dfrac{dx}{du} = \cos u$

$\therefore \quad \displaystyle\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{6}} \sqrt{1-\sin^2 u}\,\cos u\,du$

$= \displaystyle\int_0^{\frac{\pi}{6}} \cos^2 u\,du$

$= \displaystyle\int_0^{\frac{\pi}{6}} \left(\frac{1}{2} + \frac{1}{2}\cos 2u\right)du$

$= \left[\frac{1}{2}u + \frac{1}{4}\sin 2u\right]_0^{\frac{\pi}{6}}$

$= \frac{\pi}{12} + \frac{1}{4}\sin\left(\frac{\pi}{3}\right) - 0 - 0$

$= \frac{\pi}{12} + \frac{1}{4}\frac{\sqrt{3}}{2}$

$= \frac{\pi}{12} + \frac{\sqrt{3}}{8}$

18 $\displaystyle\int_0^p \left(x^3 + x\right)dx = \frac{15}{4}, \quad p > 0$

$\therefore \quad \left[\dfrac{x^4}{4} + \dfrac{x^2}{2}\right]_0^p = \frac{15}{4}$

$\therefore \quad \dfrac{p^4}{4} + \dfrac{p^2}{2} - 0 = \frac{15}{4}$

$\therefore \quad p^4 + 2p^2 = 15$

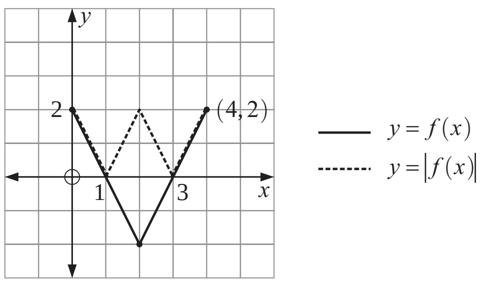
$\therefore \quad p^4 + 2p^2 - 15 = 0$

$\therefore \quad (p^2+5)(p^2-3) = 0$

$\therefore \quad p^2 = -5$ or 3

But p is real and positive, so $p = \sqrt{3}$.

19 **a**

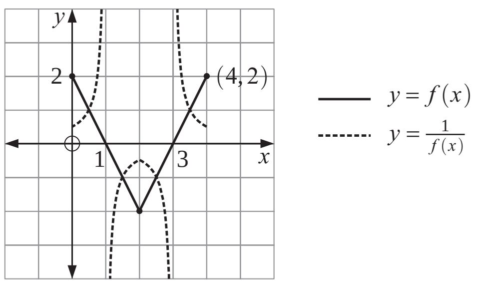


Note that $f(x)$ and $|f(x)|$ coincide for $f(x) > 0$.

b Since $f(0) = 2$ is the y-intercept of $f(x)$,

$\dfrac{1}{f(0)} = 0.5$ is the y-intercept of $\dfrac{1}{f(x)}$.

c



20 **a** The two lines will be coplanar if they have a point in common.

From $L_2: \quad x + 1 = \dfrac{y-k}{2} = \dfrac{z-1}{-2} = \mu$

we have $x = \mu - 1, \quad y = 2\mu + k, \quad z = -2\mu + 1$.

So, the lines have a point in common if:

$$\begin{cases} 3\lambda + 4 = \mu - 1 \\ \lambda + 4 = 2\mu + k \\ 2\lambda - 1 = -2\mu + 1 \end{cases}$$

$$\therefore \quad \begin{cases} 3\lambda - \mu = -5 \\ \lambda - 2\mu - k = -4 \\ 2\lambda + 2\mu = 2 \end{cases}$$

This system has solution $\lambda = -1, \ \mu = 2, \ k = -1$.

The lines are coplanar if $k = -1$ and the point of intersection is $(1, 3, -3)$.

b L_1 has direction $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ and L_2 has direction $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

$$\mathbf{p} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \quad \text{is perpendicular to both } L_1 \text{ and } L_2$$

and $\mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 1 & 2 & -2 \end{vmatrix} = -6\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}$.

c Since the point $(1, 3, -3)$ is on the line L_1, an equation of the plane normal to $\mathbf{p}$ is

$$\begin{pmatrix} -6 \\ 8 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$$

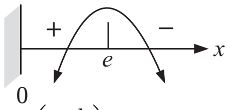
$\therefore \quad -6x + 8y + 5z = 3$

21 $f(x) = \dfrac{k \ln x}{x}, \quad k > 0, \quad x > 0$

a $$f'(x) = \frac{\left(\frac{k}{x}\right) x - k \ln x(1)}{x^2} = \frac{k - k\ln x}{x^2} = \frac{k(1 - \ln x)}{x^2}$$

$f'(x) = 0$ when $\ln x = 1$ and so $x = e$.

Sign diagram of $f'(x)$:

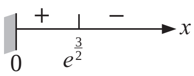


There is a local maximum at $\left(e, \dfrac{k}{e}\right)$.

b $$f''(x) = \frac{k\left(\frac{-1}{x}\right) x^2 - k(1 - \ln x)2x}{x^4} = \frac{-kx - 2kx + 2kx\ln x}{x^4} = \frac{-3k + 2k\ln x}{x^3} = \frac{k(-3 + 2\ln x)}{x^3}$$

which is 0 when $2\ln x = 3$ and so $x = e^{\frac{3}{2}}$.

Sign diagram of $f''(x)$:



There is a point of inflection at $\left(e^{\frac{3}{2}}, \dfrac{3k}{2e^{\frac{3}{2}}}\right)$.

c $$\int_1^{e^2} \frac{k\ln x}{x}\, dx = 10$$

$$\therefore \quad k\int_1^{e^2} (\ln x)\frac{1}{x}\, dx = 10$$

Let $u = \ln x$, so $\dfrac{du}{dx} = \dfrac{1}{x}$.

$$\therefore \quad k\int_1^{e^2} u\frac{du}{dx}\, dx = 10$$

When $x = 1$, $u = 0$.
When $x = e^2$, $u = 2$.

$$\therefore \quad k\int_0^{2} u\, du = 10$$

$$\therefore \quad k\left[\tfrac{1}{2}u^2\right]_0^2 = 10$$

$$\therefore \quad k(2 - 0) = 10$$

$$\therefore \quad 2k = 10$$

$$\therefore \quad k = 5$$

d We integrate by parts with $u = (\ln x)^2 \quad v' = x^{-2}$

$$u' = \frac{2\ln x}{x} \qquad v = -\frac{1}{x}$$

$$\therefore \int \frac{(\ln x)^2}{x^2}\, dx = (\ln x)^2\left(-\frac{1}{x}\right) - \int -\frac{2\ln x}{x}\left(\frac{1}{x}\right) dx = \frac{-(\ln x)^2}{x} + 2\int \frac{\ln x}{x^2}\, dx$$

We integrate by parts with $u = \ln x \quad v' = x^{-2}$

$$u' = \frac{1}{x} \qquad v = -\frac{1}{x}$$

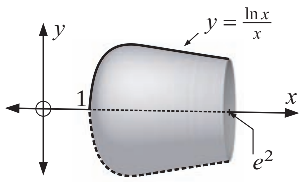
$$\therefore \int \frac{(\ln x)^2}{x^2}\, dx = \frac{-(\ln x)^2}{x} + 2\left[-\frac{\ln x}{x} - \int \frac{1}{x}\left(-\frac{1}{x}\right) dx\right]$$

$$= \frac{-(\ln x)^2}{x} - \frac{2\ln x}{x} + 2\int x^{-2}\, dx$$

$$= \frac{-(\ln x)^2}{x} - \frac{2\ln x}{x} + \frac{2x^{-1}}{-1} + c$$

$$= \frac{-(\ln x)^2}{x} - \frac{2\ln x}{x} - \frac{2}{x} + c$$

e



Volume

$$= \pi\int_1^{e^2} y^2\, dx$$

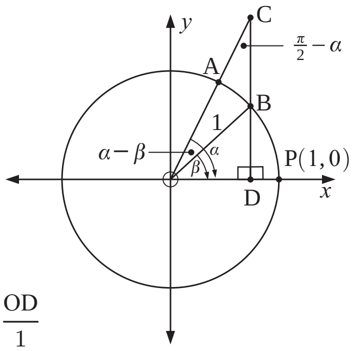
$$= \pi\int_1^{e^2} \frac{(\ln x)^2}{x^2}\, dx$$

$$= \pi\left[\frac{-(\ln x)^2}{x} - \frac{2\ln x}{x} - \frac{2}{x}\right]_1^{e^2}$$

$$= \pi\left(-\frac{4}{e^2} - \frac{4}{e^2} - \frac{2}{e^2} - (-2)\right)$$

$$= \pi\left(2 - \frac{10}{e^2}\right) \text{ units}^3$$

22



a In ΔODB:

- $\cos\beta = \dfrac{\text{OD}}{\text{OB}} = \dfrac{\text{OD}}{1}$

  $\therefore \quad \text{OD} = \cos\beta$

- $\sin\beta = \dfrac{\text{BD}}{\text{OB}} = \dfrac{\text{BD}}{1}$

  $\therefore \quad \text{BD} = \sin\beta$

b In ΔODC, $\cos\alpha = \dfrac{\text{OD}}{\text{OC}}$, so $\text{OC} = \dfrac{\text{OD}}{\cos\alpha}$

$\therefore \quad \text{OC} = \dfrac{\cos\beta}{\cos\alpha}$ {from **a**}

c $\tan\alpha = \dfrac{\text{DC}}{\text{OD}} \quad \therefore \text{DC} = \text{OD}\tan\alpha = \cos\beta\tan\alpha$

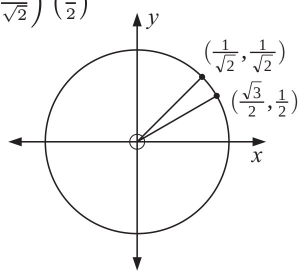
Thus $\text{BC} = \text{DC} - \text{BD} = \cos\beta\tan\alpha - \sin\beta$

d By the sine rule in ΔOBC,

$$\frac{\sin(\alpha - \beta)}{\text{BC}} = \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{1}$$

$$\therefore \quad \sin(\alpha-\beta) = \text{BC}\sin\left(\frac{\pi}{2}-\alpha\right)$$
$$= \text{BC}\cos\alpha$$
$$= (\cos\beta\tan\alpha - \sin\beta)\cos\alpha \quad \{\text{from } \mathbf{c}\}$$
$$= \left(\cos\beta\frac{\sin\alpha}{\cos\alpha} - \sin\beta\right)\cos\alpha$$
$$= \cos\beta\sin\alpha - \sin\beta\cos\alpha$$
$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

e $\sin 15^o = \sin(45^o - 30^o)$
$$= \sin 45^o\cos 30^o - \cos 45^o \sin 30^o \quad \{\text{from } \mathbf{d}\}$$
$$= \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{\sqrt{3}}{2}\right) - \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{2}\right)$$
$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$
$$= \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$



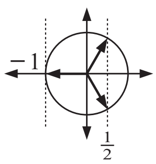
23
$$2\sin\left(x+\tfrac{\pi}{6}\right) = \sin x$$
$$\therefore \quad 2\left[\sin x\cos\tfrac{\pi}{6} + \cos x\sin\tfrac{\pi}{6}\right] = \sin x$$
$$\therefore \quad 2\sin x\left(\tfrac{\sqrt{3}}{2}\right) + 2\cos x\left(\tfrac{1}{2}\right) = \sin x$$
$$\therefore \quad \sqrt{3}\sin x + \cos x = \sin x$$
$$\therefore \quad \sin x(\sqrt{3}-1) = -\cos x$$
$$\therefore \quad \frac{\sin x}{\cos x} = \frac{-1}{\sqrt{3}-1}$$
$$\therefore \quad \tan x = \left(\frac{-1}{\sqrt{3}-1}\right)\left(\frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$$
$$= \frac{-\sqrt{3}-1}{3-1}$$
$$= -\tfrac{1}{2}(1+\sqrt{3})$$

24 $f(x) = e^{x\ln x} \qquad \therefore \quad f'(x) = e^{x\ln x}\left(1\ln x + x\left(\dfrac{1}{x}\right)\right)$
$$= e^{x\ln x}(\ln x + 1)$$
Since $e^{x\ln x} > 0$ for all x, $f'(x)$ is zero only when $\ln x = -1$.
$\therefore \quad f'(x) = 0$ when $x = e^{-1}$.

So, the x-coordinate of the stationary point is $\dfrac{1}{e}$.

25
$$y = \sin x(1+\cos x), \quad 0 \leqslant x \leqslant 2\pi$$
$$\therefore \quad \frac{dy}{dx} = \cos x(1+\cos x) + \sin x(-\sin x)$$
$$= \cos x + \cos^2 x - \sin^2 x$$
$$= \cos x + \cos^2 x - (1-\cos^2 x)$$
$$= 2\cos^2 x + \cos x - 1$$
$$= (2\cos x - 1)(\cos x + 1)$$

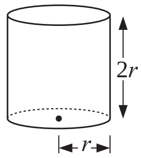
So, $\dfrac{dy}{dx} = 0$ when $\cos x = \frac{1}{2}$ or -1.
On the interval $0 \leqslant x \leqslant 2\pi$,
this means $x = \frac{\pi}{3}$, π, or $\frac{5\pi}{3}$.



When $x = \frac{\pi}{3}$, $y = \frac{\sqrt{3}}{2}\left(1+\frac{1}{2}\right) = \frac{3\sqrt{3}}{4}$.
When $x = \pi$, $y = 0$.
When $x = \frac{5\pi}{3}$, $y = -\frac{\sqrt{3}}{2}\left(1+\frac{1}{2}\right) = -\frac{3\sqrt{3}}{4}$.

The stationary points are $\left(\frac{\pi}{3}, \frac{3\sqrt{3}}{4}\right)$, $(\pi, 0)$ and $\left(\frac{5\pi}{3}, -\frac{3\sqrt{3}}{4}\right)$.

26



Surface area, $A = 2\pi r^2 + 2\pi rh$
$$\therefore \quad A = 2\pi r^2 + 2\pi r(2r)$$
$$\therefore \quad A = 6\pi r^2$$
Thus, $\dfrac{dA}{dt} = 12\pi r\dfrac{dr}{dt}$ ...... (1)

Volume $V = \pi r^2 h$
$$\therefore \quad V = \pi r^2(2r)$$
$$\therefore \quad V = 2\pi r^3$$
Thus, $\dfrac{dV}{dt} = 6\pi r^2\dfrac{dr}{dt}$ ...... (2)

Particular case: $\quad r = 5$, $\dfrac{dV}{dt} = 5\pi$
$$\therefore \quad 5\pi = 6\pi(5)^2\frac{dr}{dt}$$
$$\therefore \quad \frac{5\pi}{6\pi\times 25} = \frac{dr}{dt}$$
$$\therefore \quad \frac{dr}{dt} = \tfrac{1}{30}\text{ cms}^{-1}$$
$$\therefore \quad \frac{dA}{dt} = 12\pi(5)(\tfrac{1}{30}) = 2\pi\text{ cm}^2\text{s}^{-1}$$
So, the surface area is increasing at a rate of 2π cm^2 per second at that instant.

SOLUTIONS TO EXAMINATION PRACTICE SET 2

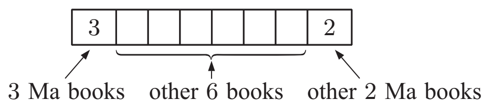
1 **a** $\frac{1}{2} < r < 1$ so the series converges.
$$u_2 = u_1 r = 6 \quad \text{and} \quad S = 49$$
$$\therefore \quad \frac{u_1}{1-r} = 49$$
$$\therefore \quad \frac{6}{r} = 49(1-r)$$
$$\therefore \quad 6 = 49r - 49r^2$$
$$\therefore \quad 49r^2 - 49r + 6 = 0$$
$$\therefore \quad (7r-1)(7r-6) = 0$$
$$\therefore \quad r = \tfrac{1}{7} \text{ or } \tfrac{6}{7}$$
But $\frac{1}{2} < r < 1$, so $r = \frac{6}{7}$

b $u_1 r = 6$ so $u_1\left(\frac{6}{7}\right) = 6$ and hence $u_1 = 7$.
So, $u_n = u_1 r^{n-1} = 7\left(\frac{6}{7}\right)^{n-1}$.

2 **a** With no restrictions, there are $8! = 40\,320$ different orderings.

b The mathematics books can be in one block in 3! ways.
This block plus the other 5 blocks can be ordered in 6! ways.
$\therefore$ total number is $3!6! = 4320$ ways.

c



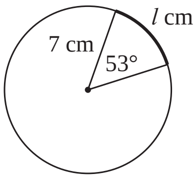
$\therefore$ there are $3\times 2\times 6! = 4320$ ways.

3
$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 5 \\ 3 & -2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$
and
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -1 \\ 4 & -2 & 1 \\ 1 & -6 & 7 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$
So,
$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 5 \\ 3 & -2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & -1 \\ 4 & -2 & 1 \\ 1 & -6 & 7 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 5 & 7 & -7 \\ 5 & -22 & 32 \\ -1 & 7 & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

$$\therefore \quad \begin{cases} y_1 = 5z_1 + 7z_2 - 7z_3 \\ y_2 = 5z_1 - 22z_2 + 32z_3 \\ y_3 = -1z_1 + 7z_2 + 2z_3 \end{cases}$$

4



a Perimeter

$$= 2r + l$$
$$= 2 \times 7 + 7\left(53 \times \tfrac{\pi}{180}\right)$$
$$\approx 20.5 \text{ cm}$$

b Area

$$= \tfrac{1}{2}r^2\theta$$
$$= \tfrac{1}{2} \times 7^2 \times \left(53 \times \tfrac{\pi}{180}\right)$$
$$\approx 22.7 \text{ cm}^2$$

5 For $\left(2x^2 + \dfrac{1}{x}\right)^9$, $\quad T_{r+1} = \binom{9}{r}(2x^2)^{9-r}\left(\dfrac{1}{x}\right)^r$

$$= \binom{9}{r} 2^{9-r} x^{18-2r} x^{-r}$$
$$= \binom{9}{r} 2^{9-r} x^{18-3r}$$

So, if we let $r = 6$, $\quad T_7 = \binom{9}{6} 2^3 x^0$

So, the constant term is $\binom{9}{6} \times 8 = 672$.

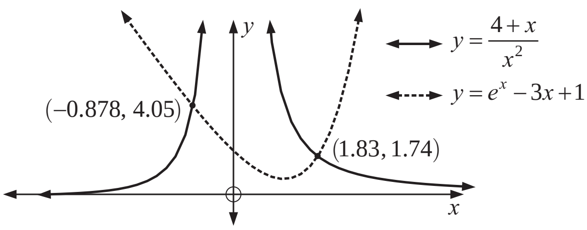
6 $(f \circ g)(x) = f(g(x)) = f(3x-2) = x + 2 \quad$ (given)

and $\quad f(9x-8) = f(3(3x-2)-2)$

$$= f(3y-2) \quad \text{letting } y = 3x-2$$
$$= y + 2$$
$$= 3x$$

7 $x^2y = 4 + x \quad \Rightarrow \quad y = \dfrac{4+x}{x^2}$ providing $x \neq 0$

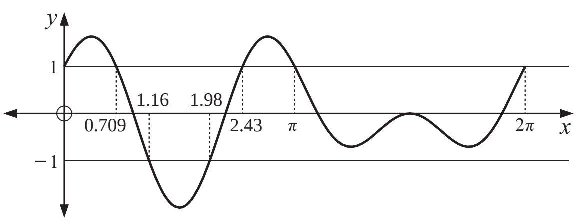
The graph of $y = \dfrac{4+x}{x^2}$ and $y = e^x - 3x + 1$ is shown.



8 $\arccos(\sin 3x + \cos 2x)$ is defined where

$-1 \leqslant \sin 3x + \cos 2x \leqslant 1$.

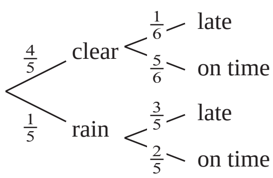
The graph of $y = \sin 3x + \cos 2x$ is obtained from technology:



$\therefore \quad x = 0, \quad 0.709 \leqslant x \leqslant 1.16, \quad 1.98 \leqslant x \leqslant 2.43,$

$\pi \leqslant x \leqslant 2\pi$

9



$$\text{P(Raining} \mid \text{Late)}$$
$$= \frac{\text{P(Raining} \cap \text{Late)}}{\text{P(Late)}}$$
$$= \frac{\frac{1}{5} \times \frac{3}{5}}{\frac{4}{5} \times \frac{1}{6} + \frac{1}{5} \times \frac{3}{5}}$$
$$= \frac{9}{19}$$

10 $X \sim \text{B}(7, p)$

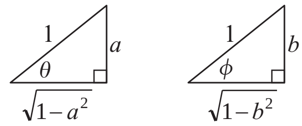
If $\text{P}(X = 4) = 0.25$ then $\binom{7}{4} p^4(1-p)^3 = 0.25$

Using technology $\quad p \approx 0.464$ or 0.674

11 Consider $\sin(\arcsin a + \arcsin b)$

Let $\arcsin a = \theta, \quad \arcsin b = \phi$

$\therefore \quad \sin\theta = a$ and $\sin\phi = b$



$$\sin(\arcsin a + \arcsin b) = \sin(\theta + \phi)$$
$$= \sin\theta\cos\phi + \cos\theta\sin\phi$$
$$= a(\sqrt{1-b^2}) + \sqrt{1-a^2}b$$
$$= a\sqrt{1-b^2} + b\sqrt{1-a^2}$$

12 **a** $\begin{pmatrix} -1 & k \\ k & 1 \end{pmatrix}^{-1} = \dfrac{1}{-1-k^2}\begin{pmatrix} 1 & -k \\ -k & -1 \end{pmatrix}$

b $-x + ky = 2k$

$kx + y = 1 - k^2$ can be written in matrix form as:

$$\begin{pmatrix} -1 & k \\ k & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2k \\ 1-k^2 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-(1+k^2)}\begin{pmatrix} 1 & -k \\ -k & -1 \end{pmatrix}\begin{pmatrix} 2k \\ 1-k^2 \end{pmatrix}$$
$$= \frac{1}{-(1+k^2)}\begin{pmatrix} 2k - k + k^3 \\ -2k^2 - 1 + k^2 \end{pmatrix}$$
$$= \frac{1}{-(1+k^2)}\begin{pmatrix} k + k^3 \\ -1 - k^2 \end{pmatrix}$$
$$= \frac{1}{-(1+k^2)}\begin{pmatrix} k(1+k^2) \\ -(1+k^2) \end{pmatrix}$$
$$= \begin{pmatrix} -k \\ 1 \end{pmatrix} \quad \text{so} \quad x = -k, \quad y = 1.$$

13 Since the vectors are perpendicular

$(\lambda\mathbf{i} + \mathbf{j} - \lambda\mathbf{k}) \bullet (3\mathbf{i} - 4\mathbf{j} + \mathbf{k}) = 0$

$\therefore \quad 3\lambda - 4 - \lambda = 0 \quad$ or $\quad \lambda = 2$

The vector $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ has length $\sqrt{2^2 + 1^2 + 2^2} = 3$

A vector of unit length parallel to

$2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is $\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$.

14 We assume the number of students in the school is large enough to use the Binomial distribution.

If X is the number of students who travel by bike, then

$X \sim \text{B}(7, \frac{2}{7})$ and $P(X = 4) = \binom{7}{4}(\frac{2}{7})^4(\frac{5}{7})^3$

$$\approx 0.0850$$

15 Let X be the number of fish suitable for sale in 20 chosen from the box. As these are selected from a box of 1000 (a large number), we assume that $X \sim \text{B}(20, 0.963)$.

a $\text{P}(X = 20) = \binom{20}{20}(0.963)^{20}$

$$\approx 0.470$$

b If exactly one is not suitable for sale, 19 are.

$\text{P}(X = 19) = \binom{20}{19}(0.963)^{19}(0.037)$

$$\approx 0.362$$

16 For $(ax+3)^5$, $\quad T_{r+1} = \binom{5}{r}(ax)^{5-r}3^r = \binom{5}{r}a^{5-r}3^r x^{5-r}$

$\therefore$ the coefficient of x^4 is $\binom{5}{1}a^4 3^1$.

For $(ax+3)^7$, $\quad T_{r+1} = \binom{7}{r}(ax)^{7-r}3^r = \binom{7}{r}a^{7-r}3^r x^{7-r}$

$\therefore$ the coefficient of x^5 is $\binom{7}{2}a^5 3^2$.

Thus $\binom{5}{1}a^4 3^1 = \binom{7}{2}a^5 3^2$ and so $15a^4 = 21 \times 9a^5$

$\therefore$ as $a \neq 0$, $a = \dfrac{15}{21 \times 9} = \frac{5}{63}$.

17 **a**
$$\begin{aligned}&(g \circ f)(4)\\ &= g(f(4))\\ &= g\left(\tfrac{1}{4+5}\right)\\ &= g\left(\tfrac{1}{9}\right)\\ &= 3\left(\tfrac{1}{9}\right)\\ &= \tfrac{1}{3}\end{aligned}$$

b g is $y = 3x$

g^{-1} is $x = 3y$

$\therefore\ y = \dfrac{x}{3}$

$g^{-1}(x) = \dfrac{x}{3}$

c Domain of g^{-1} is $x \in \mathbb{R}$.

18 **a** **A** is non-singular if $\det \mathbf{A} \neq 0$, and

$$\begin{aligned}\det \mathbf{A} &= \begin{vmatrix} 1-k & -2 & -1 \\ 1 & -1 & -2 \\ 1 & k & -1 \end{vmatrix}\\ &= (1-k)\begin{vmatrix} -1 & -2 \\ k & -1 \end{vmatrix} - -2\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} + -1\begin{vmatrix} 1 & -1 \\ 1 & k \end{vmatrix}\\ &= (1-k)(1+2k) + 2(1) - 1(k+1)\\ &= 1 + 2k - k - 2k^2 + 2 - k - 1\\ &= 2 - 2k^2\\ &= 2(1+k)(1-k)\end{aligned}$$

So, **A** is non-singular for $k \neq \pm 1$.

b If $k = 0$, $\mathbf{A} = \begin{pmatrix} 1 & -2 & -1 \\ 1 & -1 & -2 \\ 1 & 0 & -1 \end{pmatrix}$

$\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{2} & -1 & \frac{3}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}$ {using technology}

c Given $\begin{cases} x - 2y - z = 2 \\ x - y - 2z = -1 \\ x - z = 2 \end{cases}$ we write $\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$.

Thus $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -1 & \frac{3}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

$\therefore\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$

Thus, the point of intersection is $(5, 0, 3)$.

d The line has equations: $\dfrac{x+3}{2} = y + 4 = \dfrac{z-7}{-1}$

If $x = 5$, $\dfrac{x+3}{2} = \dfrac{5+3}{2} = 4$

If $y = 0$, $y + 4 = 4$

If $z = 3$, $\dfrac{z-7}{-1} = \dfrac{-4}{-1} = 4$

So, $(5, 0, 3)$ lies on the line.

e Let θ be the acute angle between the line and the normal to the plane.

Now the direction vector of the line is $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

and the direction vector of the plane's normal is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

$\therefore\ \cos\theta = \dfrac{|2 - 2 - 1|}{\sqrt{6}\sqrt{6}} = \frac{1}{6}$ and so $\theta \approx 80.41^o$.

The required angle is $90^o - 80.41^o \approx 9.59^o$.

19 **a** Let X be the measured speed of a car travelling at 60 km h^{-1}.
The error in reading the speed is $E = X - 60$.
So, $X = 60 + E$.

b Since $E \sim \mathrm{N}(0, \sigma^2)$ $X \sim \mathrm{N}(60, \sigma^2)$

$$\begin{aligned}\mathrm{P}(X \geqslant 65) &= 0.01\\ \therefore\ \mathrm{P}(X \leqslant 65) &= 0.99\\ \therefore\ \mathrm{P}\left(\frac{X-60}{\sigma} \leqslant \frac{65-60}{\sigma}\right) &= 0.99\\ \therefore\ \mathrm{P}\left(Z \leqslant \frac{5}{\sigma}\right) &= 0.99\\ \therefore\ \frac{5}{\sigma} &\approx 2.33\\ \therefore\ \sigma &\approx 2.15\end{aligned}$$

20
$$\begin{aligned}\text{Since}\quad \mathbf{a} + \mathbf{b} + \mathbf{c} &= \mathbf{0}\\ \mathbf{a} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) &= \mathbf{0}\\ \therefore\ \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} &= \mathbf{0} \quad \{\text{since } \mathbf{a} \times \mathbf{a} = \mathbf{0}\}\\ \therefore\ \mathbf{a} \times \mathbf{b} &= -(\mathbf{a} \times \mathbf{c}) \quad ......(1)\end{aligned}$$

$$\begin{aligned}\text{Also}\quad (\mathbf{a} + \mathbf{b} + \mathbf{c}) \times \mathbf{c} &= \mathbf{0}\\ \therefore\ \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} &= \mathbf{0} \quad \{\text{since } \mathbf{c} \times \mathbf{c} = \mathbf{0}\}\\ \therefore\ \mathbf{b} \times \mathbf{c} &= -(\mathbf{a} \times \mathbf{c}) \quad ......(2)\end{aligned}$$

From (1) and (2) $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c}$

21 Let X be the diameter of a disc, then $X \sim \mathrm{N}(73, 1.1^2)$
Hence, $\mathrm{P}(X > 75) \approx 0.0345$.

22 $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$
Since A and B are independent $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$
Hence, letting $\mathrm{P}(B) = p$, $0.75 = 0.35 + p - 0.35p$
So, $0.65p = 0.40$
$\therefore\ p = \frac{40}{65} = \frac{8}{13}$

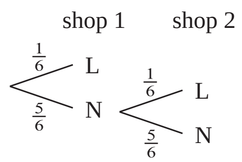
23 Let X be the number of blonde children. Then $X \sim \mathrm{B}(5, \frac{1}{7})$.

a $\mathrm{E}(X) = np = 5 \times \frac{1}{7} = \frac{5}{7}$
Expected number of blonde children is $\frac{5}{7}$.

b
$$\begin{aligned}&\mathrm{P}(X = 3)\\ &= \tbinom{5}{3}\left(\tfrac{1}{7}\right)^3\left(\tfrac{6}{7}\right)^2\\ &\approx 0.0214\end{aligned}$$

c
$$\begin{aligned}&\mathrm{P}(X > 3)\\ &= 1 - \mathrm{P}(X \leqslant 3)\\ &\approx 1 - 0.99816\\ &\approx 0.00184\end{aligned}$$

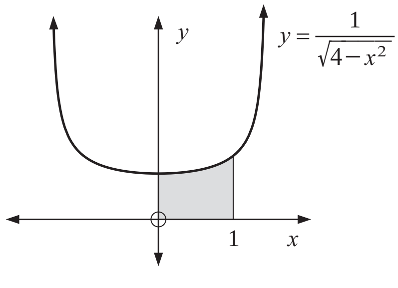
24 A tree diagram shows the situation.



Probability she leaves it in shop 1 given it is missing is

$= \dfrac{\text{P(shop 1 and missing)}}{\text{P(missing)}} = \dfrac{\frac{1}{6}}{\frac{5}{36} + \frac{1}{6}} = \frac{6}{11}$

25



a Area $= \displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\arcsin\left(\frac{x}{2}\right)\right]_0^1$

$= \arcsin\left(\frac{1}{2}\right) - \arcsin(0)$

$= \frac{\pi}{6} - 0$

$= \frac{\pi}{6}$ units^2

b Volume $= \pi \displaystyle\int_0^1 \frac{1}{4-x^2}\,dx$

≈ 0.863 units^3 {using technology}

For the exact value

Let $\dfrac{1}{4-x^2} = \dfrac{A}{2+x} + \dfrac{B}{2-x}$

$\therefore \quad A(2-x) + B(2+x) = 1$

If $x = 2, \quad 4B = 1 \quad \therefore \quad B = \frac{1}{4}$

If $x = -2, \quad 4A = 1 \quad \therefore \quad A = \frac{1}{4}$

$\therefore \quad$ Volume $= \pi \displaystyle\int_0^1 \left(\frac{\frac{1}{4}}{x+2} + \frac{\frac{1}{4}}{2-x}\right) dx$

$= \frac{1}{4}\pi \left[\ln|x+2| - \ln|2-x|\right]_0^1$

$= \frac{\pi}{4}(\ln 3 - \ln 1 - \ln 2 + \ln 2)$

$= \dfrac{\pi \ln 3}{4}$ units^3

26 **a** Since $\frac{2}{7} + \frac{1}{7} + \frac{3}{14} + \frac{1}{14} + \frac{1}{7} + y = 1$

$\frac{6}{7} + y = 1 \quad \therefore \quad y = \frac{1}{7}$

b $\text{E}(X)$

$= 1 \times \frac{2}{7} + 2 \times \frac{1}{7} + 3 \times \frac{3}{14} + 4 \times \frac{1}{14} + 5 \times \frac{1}{7} + 6 \times \frac{1}{7}$

$= \frac{1}{14}(4 + 4 + 9 + 4 + 10 + 12) = \frac{43}{14}$

$\text{Var}(X)$

$= \text{E}(X^2) - (\text{E}(X))^2$

$= 1^2(\frac{2}{7}) + 2^2(\frac{1}{7}) + 3^2(\frac{3}{14}) + 4^2(\frac{1}{14}) + 5^2(\frac{1}{7}) + 6^2(\frac{1}{7})$

$- \left(\frac{43}{14}\right)^2$

$= \frac{177}{14} - \left(\frac{43}{14}\right)^2 = \frac{629}{196}$

c If the die is tossed many times we expect the mean value of the tosses to be $\frac{43}{14}$ with standard deviation of $\sqrt{\frac{629}{196}}$.

SOLUTIONS TO EXAMINATION PRACTICE SET 3

1 **a** $\displaystyle\sum_{r=1}^{3} (2r + 2^r) = (2+2) + (4+4) + (6+8) = 26$

b $\displaystyle\sum_{r=1}^{n} (2r + 2^r)$

$= 2 + 2^1 + 4 + 2^2 + 6 + 2^3 + 8 + 2^4 + ... + 2n + 2^n$

$= (2 + 4 + 6 + 8 + ... + 2n) + (2 + 2^2 + 2^4 + ... + 2^n)$

$= \underbrace{\frac{n}{2}(2+2n)}_{\text{arithmetic sum}} + \underbrace{\frac{2(2^n - 1)}{2-1}}_{\text{geometric sum}}$

$= n(n+1) + 2(2^n - 1)$

2 $\log_a \sqrt{72} = \log_a (2^3 3^2)^{\frac{1}{2}}$

$= \log_a (2^{\frac{3}{2}} 3^1)$

$= \frac{3}{2}\log_a 2 + \log_a 3$

$= \frac{3}{2}b + c$

3 $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\cos^2 x + \tan^2 x\right) dx$

$= \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(2x) + \sec^2 x - 1\right) dx$

$= \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\tfrac{1}{2}\cos(2x) + \sec^2 x - \tfrac{1}{2}\right) dx$

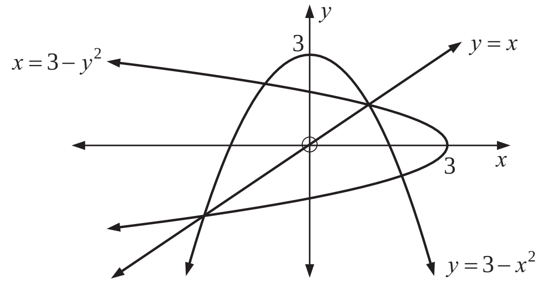
$= \left[\frac{1}{4}\sin(2x) + \tan x - \frac{1}{2}x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \frac{1}{4}\sin\left(\frac{2\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right) - \frac{\pi}{6} - \frac{1}{4}\sin\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{6}\right) + \frac{\pi}{12}$

$= \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right) + \sqrt{3} - \frac{\pi}{6} - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{\sqrt{3}} + \frac{\pi}{12}$

$= \sqrt{3} - \frac{1}{\sqrt{3}} - \frac{\pi}{12}$

4 **a**



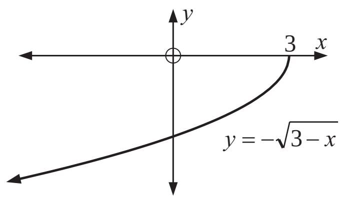
b The inverse of $y = f(x)$ is not a function since for one value of x there may be more than one value of y.
For example, when $x = 0$, y has two values, $y = -3$ as well as $y = 3$.

c The function f is $y = -x^2 + 3$, $x \leqslant 0$, and so the function f^{-1} is $x = -y^2 + 3$, $y \leqslant 0$

$\therefore \quad y^2 = 3 - x$

$\therefore \quad y = \pm\sqrt{3-x}$

But $y \leqslant 0$ and so $y = -\sqrt{3-x}$.



5 $f(x) = 2x^3 - x^2 - 8x - 5$

$f(-1) = 0$ hence $(x+1)$ is a factor.

$\therefore \quad 2x^3 - x^2 - 8x - 5 = (x+1)(2x^2 + ax - 5)$

Equating coefficients of x^2, $\quad -1 = 2 + a \quad \therefore \quad a = -3$

So, $\quad 2x^3 - x^2 - 8x - 5$

$= (x+1)(2x^2 - 3x - 5)$

$= (x+1)(x+1)(2x-5)$

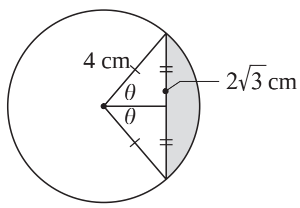
$= (x+1)^2(2x-5)$

$f(x) > 0$ if $2x - 5 > 0, x \neq -1$

{since $(x+1)^2 > 0$ for all $x \neq -1$}

$\therefore \ f(x) > 0$ if $x > \frac{5}{2}$.

6



$\sin\theta = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

$\therefore \quad \theta = \frac{\pi}{3}$

Area $= \frac{1}{2}r^2 2\theta - \frac{1}{2}r^2\sin(2\theta)$

$= \frac{1}{2} \times r^2\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$

$= 8\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$ cm^2

$= \left(\frac{16\pi}{3} - 4\sqrt{3}\right)$ cm^2