Analysis & Approaches HL

This table records some of the elements of the Analysis & Approaches HL book which are particularly "IB", or which are interesting "features". They are definitely things to look out for, but please do not consider this an exhaustive list.

	Page	Topic link	Subject link	International link	Cultural link	Historic link	TOK link	Comments

Chapter 1: Further Trigonometry

Exercise 1A q9	19							Simple question which gives geometric meaning to the reciprocal trigonometric ratios for acute angles.
Historical note	19-20		Astronomy	Ancient Greece, India Europe		Hipparchus, Ptolemy, Aryabhata, Rheticus, Copernicus		The development of trigonometry involved contributions from many people groups, over several millennia. This could be discussed alongside the work of the Chinese astronomer Li Chunfeng (see Core Topics ch7 p158-159).
Investigation 1	20-21	Functions						This important Investigation provides solid grounds for why the rigours of functional notation and properties are needed, in particular domain and range. We apply the idea of an inverse function to the trigonometric functions already studied on restricted domains.
Activity 1	23	Continued fractions				Carl Friedrich Gauss		
Exercise 1D q14a	30							Derivation of the important identities for (cos x)^2 and (sin x)^2 used in their integration.
Activity 2	31							Parametric equations are a fun opportunity for exploration, even more so if returned later with complex numbers. Consider the complex function n(t) = x(t) + i * y(t).
Activity 3	38	Series						Truncation of an infinite trigonometric series allows us to predict the graph of the infinite series. This highlights how a piecewise function may actually be described exactly using an infinite series.

Chapter 2: Exponential Functions

Investigation 1	49-50	Transformation of functions						Builds on from the transformation of functions chapter to give conceptual understanding of the general exponential function.
Investigation 2	61	Compound interest						This investigation gives pre-limits derivation of the natural exponential e by considering compound interest compounding at a faster and faster rate.

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Historical note	62	Continued fractions				Jacob Bernoulli, Leonhard Euler		Exact representations of the irrational number e.

Chapter 3: Logarithms

Theory of Knowledge	78-79		Physics	Scotland		John Napier	Nature of mathematics	Do we invent or discover mathematics? Is mathematics a collaborative effort? Why is pure mathematics important?
Investigation 3	90-92		Music, Physics, Geography, Chemistry					Logarithmic scales are widely used to understand the real world. In this Investigation we explore: musical notes, the Richter scale for earthquakes, the pH scale for acidity, and the decibel scale for sound intensity.

Chapter 4: Introduction to Complex Numbers

Opening Problem	98	Quadratic equations						Invites students to consider whether the square root of a negative number could have meaning. If so, do the solutions to a quadratic equation with negative discriminant have the same sum and product properties as the solutions to a quadratic equation with positive discriminant?
Historical note	98-99			Roman Egypt, Italy		Heron of Alexandria, Gerolamo Cardano, Rafael Bombelli		It took nearly 1500 years from when the idea that the square root of a negative number may have meaning, to the definition of i = sqrt(-1).
Historical note	105			Germany		Carl Friedrich Gauss		Carries on the narrative from the previous Historical note.

Chapter 5: Real Polynomials

Activity 1	113							The shift-expand-shift procedure writing a polynomial in terms of a simple linear expression is extremely useful in calculus, in particular for study of Taylor series (of which Maclaurin series later in the course are the special case).
Activity 2	122							Synthetic division is a useful tool for writing polynomial division more concisely.
Activity 3	124							The use of a grid to perform division by a quadratic is significantly quicker than using long division.

Chapter 6: Further Functions

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Example 14	169							In step 4, the choice of substitutions for x enable us to immediately evaluate the coefficients. This is far more efficient than expanding the RHS and then solving simultaneous equations. However, it is surprising how few people choose the quicker substitution method.

Chapter 7: Counting

Discussion	181						Definitions	Do we define things simply for our own convenience?

Chapter 8: The Binomial Theorem

Investigation 1	194-195							Connects the binomial expansion to Pascal's triangle.
Investigation 2	198							Explains the formula for the binomial coefficient using the combinations from the previous chapter.
Historical note	202					Sir Isaac Newton		Introduces the idea of a binomial expansion for rational powers.
Theory	203							The idea of convergence is essential for any meaningful discussion of series representations. In the absence of a formal ratio test in this course, we can still compare with the infinite geometric series already studied to give intuitive understanding of the convergence of the binomial expansion for rational powers. This will be revisited later in the related Maclaurin series.

Chapter 9: Reasoning and Proof

Exercise 9A q6 Review Set 9B q15	213 232					Peter Wason		Classic problems of logic.
Exercise 9B q9 Exercise 9C q7 Review Set 9B q8	216 219 232							Identifying incorrect steps in proofs is extremely effective in developing conceptual understanding.
Historical note	216			England		Charles Dodgson	Logic	
Exercise 9C q8	219							Students should recognise the difference between deduction and equivalence. This question explores an example used incorrectly in the syllabus (2019).
Theory of Knowledge	219						Definitions, Proof	How do our definitions and our use of words affect proofs and our mathematical understanding? When we assess algebraic solutions, we may allow expressions which are equal and are equivalent (to a given level of simplicity) but which are not the same as the listed solution.
Theory of Knowledge	221-222					Kurt Gödel, Pierre de Fermat	Axioms	What is an axiom? Why are axioms necessary in mathematics?

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Exercise 9E q8	223							A simply posed question but very difficult to conveniently annotate or explain a solution. It could be solved by listing many, many cases. However, even in proof by exhaustion, we should aim to be concise and specific.
Section G	225-227							Proof by contrapositive has been included because it is so often confused with proof by contradiction.
Historical note	227			England, India		Godfrey Harold Hardy, John Littlewood, Srinivasa Ramanujan		The relationship between these famous 20th century mathematicians was recently featured in the film "The Man Who Knew Infinity".
Discussion	230					Carl Gauss		Similarities and differences between different forms of proof.

Chapter 10: Proof by Mathematical Induction

Opening Problem	234			Europe		Blaise Pascal, Yang Hui	Parallel development	
Historical note	234			Europe, India, Persia, China		Leonhard Euler, Francesco Maurolico, Blaise Pascal		
Exercise 10A q5	237							Highlights the danger of assuming a pattern will continue in a particular form, and thus the importance of proof.
Section B	237-249							This extremely comprehensive section on Mathematical Induction highlights how a method or form of proof can be applied to topics all across the subject. We consider divisibility, functions, sequences and series, inequalities, and geometry.
Activity	250-251					Pierre de Fermat		Fermat's Method of Infinite Descent was a forerunner to Mathematical Induction. In some cases it provides exceptionally elegant proofs.

Chapter 11: Linear Algebra

Theory and Discussion Theory	259 261-263							Identification and understanding of form is key to solving simultaneous linear systems.

Chapter 12: Vectors

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Investigation 2	299 (link)							The unique representation of an n-dimensional vector as a linear combination of n mutually perpendicular vectors is considered for n = 2, 3. This idea is fundamental to the unique specification of a point in the Cartesian coordinate system. The logical ideas expand readily to the vector equation of a line.
Discussion	301-302							In 2 dimensions, all vectors through a point which are perpendicular to the non-zero vector v, are parallel to one another. In 3 dimensions, the vectors through a point which are perpendicular to the non-zero vector w, are not all parallel to one another, and in fact form a plane.
Investigation 3	310-311							This investigation provides the matrix determinant background to the vector cross product formula. Subsequent worked examples are written in terms of both this matrix determinant definition and the formula booklet representation.

Chapter 13: Vector Applications

Opening Problem	324							Builds on from the Discussion in the previous Chapter (p301-302) to lead students to the information necessary for defining a plane.
Theory of Knowledge	337-338			Europe		August Möbius, Sir William Hamilton, Josiah Willard Gibbs, Oliver Heaviside	Parallel development	
Investigation 2	345-346							Builds on from the introduction to linear combinations in Chapter 12, Investigation 2.
Investigation 4	352							In Exercise 13G q21 we derive the formula for the shortest distance between a point and a line. In this Investigation we derive related formulas for the shortest distance between a line and a plane, and between two skew lines.

Chapter 14: Complex numbers

Opening Problem	368							Builds directly from the Opening Problem from Introduction to Complex Numbers chapter, now motivating a complex plane using vectors.
Activity 1	377	Vector geometry, mathematical induction						This Activity on the Triangle Inequality links complex numbers, vector geometry, and mathematical induction.

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Activity 2	377-378							This Activity on locus shows how complex number representation can be more efficient than the real Cartesian plane for describing sets of points.
Historical note	386					Leonhard Euler	What is beauty?	Euler's "beautiful" equation links the three great constants of mathematics: exponential e, imaginary i, and the ratio π of a circle's circumference to its diameter.
Investigation	386-387	Compound interest						Having previously used the idea of continuously compounding interest to motivate exponential e, we now consider the idea of "imaginary" compound growth. By tracing the movement of a number in the complex plane, we deduce that multiplication by $e^{i\theta}$ produces an anticlockwise rotation in the Argand plane through angle θ. This is compared with multiplication by the real e^r which produces an enlargement.

Chapter 15: Limits

Opening Problem	400							We choose to use a very simple sequence to introduce limits. Students should well understand that as the term increases, the sequence gets closer and closer to 1/3.
Historical note	400			Europe		Archimedes, John Wallis, Augustin-Louis Cauchy, Bernard Bolzano, Karl Weierstrass	Proof	As far as Archimedes was concerned, his method of infinitesimals constituted proof of his results. Some 2000 years later, limits were formally defined, adding an extra layer of rigour to the proofs of Archimedes. Will there be a point where what we regard today as "proven" will be called into question, and re-examined in the light of new theory, to be subtly but importantly clarified or corrected?
Theory of Knowledge	403		Physics	Ancient Greece		Zeno of Elea	Paradoxes	
Exercise 15B q12	406							This is a very surprising function!
Investigation	409	Area, Functions						Neat area proof for the limit of $\sin(x) / x$ as $x \to 0$. Students should be aware of the need for proving the limit not just as $x \to 0+$ but also as $x \to 0-$.
Exercise 15E q7	412							Students should be aware of the limitations of technology.
Review Set 15A q5 Review Set 15B q4	413 414	Area						Further proofs of limiting values based on area.

Chapter 16: Introduction to Differential Calculus

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Historical note	417			Ancient Egypt, Ancient Greece, Europe		Democritus, Eudoxus, Archimedes, Johann Bernoulli, Isaac Barrow		
Discussion	424							Building on ideas of continuity from the previous chapter, we discuss the existence of the limit defined as the gradient of a tangent to a curve, hence leading students towards differentiability. This is valuable for getting a conceptual understanding of what could cause a lack of differentiability before they see the formal definitions in section F.
Historical note	432	Series		Italy, Germany		Bernard Bolzano, Karl Weierstrass		

Chapter 17: Rules of Differentiation

Opening Problem	436	Transformation of functions						The transformation of functions previously studied can give clues to the relationships between derivative functions.
Investigation 1	436-437	Binomial expansion						Uses first principles and the binomial expansion with integer powers to deduce the derivative of terms of the form a*x^n where n is a positive integer.
Investigation 2	442							Leads to the Chain rule.
Investigation 3	444-445							Leads to the Product rule.
Investigation 4 Investigation 5	449 450							Leads to the derivative of e^x.
Investigation 6	454							Leads to the derivative of ln x.
Investigation 7	457-458							Leads to the derivatives of sin x and cos x.
Exercise 17G.1 q11	460							Calculus derivation of Euler's formula.

Chapter 18: Properties of Curves

Exercise 18H q11	508	Compound interest						Use of l'Hôpital's rule to illustrate the connection between the limit we saw in compound interest and exponential e.
Historical note	508			Switzerland, France		Johann Bernoulli, Marquis Guillaume de l'Hôpital	Ethics	Is it ethical to buy someone's discovery?
Review Set 18A q35	512	Quadratic functions						Uses the shift-expand-shift principle to explore the tangent to a quadratic.

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Review Set 18B q35	515-516		Physics					Considers the focus-directrix definition of a parabola, and the reflective property that any vertical rays will be reflected through the focus.

Chapter 19: Applications of Differentiation

Exercise 19C q20	537		Physics			Pierre de Fermat		Calculus derivation of the Law of Reflection and Snell's Law of Refraction
Theory of Knowledge	537		Physics			Ibn Sahl, Willebrord Snellius, René Descartes		
Activity	537 (link)		Graphic Design, Engineering					Cubic splines are a popular and useful modelling tool.

Chapter 20: Introduction to Integration

Opening Problem	544		Physics			Archimedes		We begin the study of integration by following its historical development.
Investigation 1	546	Series, Limits						Using the same series formula as used in the Core Topics investigation deriving the volume of a tapered solid formula, we prove Archimedes' result for the area under $y = x^2$ on the interval $0 < x < 1$.
Historical note	547			Italy		Bonaventura Cavalieri		
Historical note	548					Sir Isaac Newton, Gottfried Wilhelm Leibniz, Bernhard Riemann	Parallel development	The progression from Archimedes to modern calculus was only possible with the introduction of limits.
Exercise 20B q3	549							Links to the standard normal deviation and the proportion of data within 3 standard deviations of the mean.
Investigation 2	551	Geometric sequences and series				Pierre de Fermat		Uses the formula for an infinite geometric series to derive the formula for the definite integral under $y = x^k$, k an integer not -1, on the interval $0 < x < b$.

Chapter 21: Techniques for Integration

Exercise 21A	563-565							This exercise is built as an Investigation leading to the rules of integration.

Chapter 22: Definite Integrals

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Activity 3	618-619	Probability				Georges-Louis Leclerc, Comte de Buffon		First historical application of calculus to probability.
Section I	620-622							There is no mention (either inclusion or exclusion) of improper integrals in the syllabus. However, given we have already studied l'Hôpital's rule and the clear link to understanding continuous statistical distributions, this section has been written with that focus.
Exercise 22I q7	622					Evangelista Torricelli	Paradoxes	How can an object have finite volume but infinite surface area?
Review Set 22B q24	628							Derives a function giving the values of n! for any positive integer n.

Chapter 23: Kinematics

Discussion	631	Vectors	Physics					From the outset, students can discuss the terminology they have for motion, and how the physics and mathematics relate.
Investigation	647-649	Vectors	Physics	England, Italy		Galileo Galilei	Ethics	The study of projectile motion was driven by its applications in war. Does this negate the virtue of its study?

Chapter 24: Maclaurin Series

Opening Problem	654							The tangent to a curve at a particular point can be described as the straight line which best approximates the curve at that point. In this Opening Problem we develop this idea to a best approximating quadratic, cubic, and so on to a best approximating polynomial, which is the Maclaurin series.
Historical note	654			Ancient Greece, China, India, Scotland		Zeno of Elea, Plato, Liu Hui, Mādhava of Sangamagrāma, James Gregory, Brook Taylor	Parallel development	
Investigation 1	655							Introduces the Taylor series of which the Maclaurin series is a special case. Several important concepts are covered including where information about the function comes from, and whether a Taylor or Maclaurin series will be an exact representation of every curve for all x. This leads into the idea of convergence later.

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Investigation 2	658					Leonhard Euler		Takes a deeper look at the integral of x^k, showing that in the limit as k tends to -1, the usual rule for the integral in fact tends to ln x.
Exercise 24B q2	660	Binomial expansion						Links the binomial theorem for rational powers to the Maclaurin series for the same expression.
Exercise 24D q2	664							Introduces the hyperbolic trigonometric functions cosh and sinh.
Exercise 24D q3	664							Uses Maclaurin series to derive Euler's formula for e^iθ.
Exercise 24G q7	670					Leonhard Euler		Follows Euler's proof for the infinite series expansion of (π^2)/6.

Chapter 25: Differential Equations

Historical note	694		Biology			Pierre François Verhulst, Raymond Pearl, Lowell Reed		
Activity 1	700-701	Complex numbers	Physics					As well as being a wonderful real-world example familiar to most students, simple harmonic motion ties together several mathematical themes. Euler's formula for e^iθ provides the vital link between exponential and trigonometric functions.
Activity 2	701 (link)			France		Pierre-Simon (Marquis de Laplace)		Provides an introduction to Laplace transforms, including the solution for the simple harmonic motion differential equation.

Chapter 26: Bivariate Statistics

Historical note	713					Karl Pearson, Sir Francis Galton		
Activity 2	726			England		Francis Anscombe		
Theory of Knowledge	727-728		Biology, Environmental Science	Japan, Global			Modelling	
Theory of Knowledge	732				Equality and Discrimination		Equality	

Chapter 27: Discrete Random Variables

Exercise 27B q17	745	Maclaurin series						
Activity	750				Game strategy			

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Investigation 2 Investigation 3	760 763							Use of technology to investigate the binomial distribution.

Chapter 28: Continuous Random Variables

Investigation 1	771	Cumulative frequency graphs						Students need to understand the difference between a probability mass function for discrete variables, and a probability density function for continuous variables.
Exercise 28B q12	777	Improper integrals, Queuing theory						
Exercise 28B q15	778							Considers non-linear transformations of a continuous random variable.
Historical note	780					Carl Friedrich Gauss		
Investigation 2	780	Calculus						Investigates the normal curve using differential calculus.
Exercise 28C.2 q4	781					Pierre-Simon (Marquis de Laplace)		
Exercise 28C.2 q5	781-782	Maclaurin series						The moment of a random variable, moment generating functions.
Investigation 5	798							The normal approximation to the binomial distribution.