Applications & Interpretation HL

This table records some of the elements of the Applications & Interpretation HL book which are particularly “IB”, or which are interesting “features”. They are definitely things to look out for, but please do not consider this an exhaustive list.

	Page	Topic link	Subject link	International link	Cultural link	Historic link	TOK link	Comments

Chapter 1: Exponentials

Investigation 1	21-22	Transformation of functions						Builds on from the transformation of functions chapter to give conceptual understanding of the general exponential function.
Investigation 2	35	Compound interest						This investigation gives pre-limits derivation of the natural exponential e by considering compound interest compounding at a faster and faster rate.
Historical note	38	Continued fractions				Jacob Bernoulli, Leonhard Euler		Exact representations of the irrational number e.

Chapter 2: Logarithms

Opening Problem	46			England		William Oughtred	Technology	
Theory of Knowledge	55-56		Astronomy	Europe		John Napier, Johannes Kepler, Sir Isaac Newton, Pierre-Simon (Marquis de Laplace)	Nature of mathematics	Do we invent or discover mathematics? Is mathematics a collaborative effort? Why is pure mathematics important?
Section E	59-62		Music, Physics, Geography, Chemistry					Logarithmic scales are widely used to understand the real world. We include musical notes, the Richter scale for earthquakes, the pH scale for acidity, and the Krubein phi scale for particle size.
Activity	63		Astronomy	England		Norman Pogson		The brightness of stars is measured in terms of apparent magnitude and absolute magnitude.
Review Set 2B q10	65		Physics					Continues the section E theme, this time with the decibel scale for sound intensity.

Chapter 3: Approximations and Error

Chapter 4: Loans and Annuities

Historical note	78			Italy		Banking		
Activity 1 Discussion	82-83 84		Personal finance					Understanding the effect of interest rates and loan duration is essential for future life planning.

	Page	Topic link	Subject link	International link	Cultural link	Historic link	TOK link	Comments
Activity 2	83-84	Series	Personal finance					Provides an algebraic development of the loan repayments formula.
Activity 3 Activity 4	87 87-88		Personal finance					Perpetuities and growing annuities are special types of annuities.

Chapter 5: Modelling

Theory	92-93						Scientific Method	Is the modelling cycle the mathematical equivalent of the scientific method?
Activity	96							Enables students to practice the construction of a mathematical model, including assumptions and simplification.
Investigation	96-97		Astronomy	Ancient Rome, England, Italy		Julius Caesar, Roger Bacon, Pope Gregory XIII		Calendars
Theory of Knowledge	97-98		Astronomy, Physics	Europe		Nicolaus Copernicus, Johannes Kepler, Sir Isaac Newton, George Box	Modelling	Historic models such as a clockwork orrery for the solar system provide insight into physical processes, but have never been exact. Can an imperfect model be useful?

Chapter 6: Direct and Inverse Variation

Theory of Knowledge	121		Physics	England		William Thomson (Lord Kelvin)	Measurement	Are some scales of measurement more natural or intuitive than others?

Chapter 7: Bivariate Statistics

Historical note	141					Karl Pearson, Sir Francis Galton		
Activity 2	157	Summation notation, quadratic theory						Optimisation using quadratic theory is useful in surprisingly many contexts, providing ample justification for its study. In this case, we use it to derive a formula for a linear model through the origin which minimises the sum of the least squares residuals.
Discussion	158							Strangely, the A&I HL course requests understanding of the difference between the regression line of y against x, and the regression line of x against y, but the regression line of x against y is not calculated in this course. It is instead calculated in the A&A HL course.
Theory of Knowledge	158-160		Biology, Environmental Science	Japan, Global			Modelling	

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Activity 4	168-169			England		Francis Anscombe		

Chapter 8: Non-linear Modelling

Exercise 8A q3	178	Series						Considers the convergence of an infinite harmonic series by applying a logarithmic model.
Activity 1	185-186		Physics					Models the period of a pendulum. The pendulum will become a recurring theme, to be revisited in differential equations.
Activity 2	186 (link)		Medicine, Physics, Biology					Non-linear models include the surge model for drug efficacy, terminal velocity for a falling object, and logistic model for population growth or the spread of a rumour. Population growth will become a recurring theme, to be revisited in integral calculus and differential equations.
Activity 3	187							Uses spreadsheets to give an understanding of what non-linear regression means.
Discussion	188							Compares true non-linear regression with the models produced by a calculator.

Chapter 9: Vectors

Investigation 1	220 (link)							Any vector in the Cartesian plane can be uniquely written as a linear combination of any two non-zero, non-parallel vectors.
Discussion	222							In 2 dimensions, all vectors through a point which are perpendicular to the non-zero vector v, are parallel to one another. In 3 dimensions, the vectors through a point which are perpendicular to the non-zero vector w, are not all parallel to one another, and in fact form a plane.
Investigation 2	228							This investigation provides the matrix determinant background to the vector cross product formula. There is no point in waiting for the coming matrix work, as it does not include 3x3 determinants. Subsequent worked examples are written in terms of both this matrix determinant definition and the formula booklet representation.

Chapter 10: Vector Applications

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Theory of Knowledge	260-261			Europe		August Möbius, Sir William Hamilton, Josiah Willard Gibbs, Oliver Heaviside	Parallel development	

Chapter 11: Complex Numbers

Opening Problem	266	Quadratic equations						Invites students to consider whether the square root of a negative number could have meaning. If so, do the solutions to a quadratic equation with negative discriminant have the same sum and product properties as the solutions to a quadratic equation with positive discriminant?
Historical note	267			Roman Egypt, Italy		Heron of Alexandria, Gerolamo Cardano, Rafael Bombelli		It took nearly 1500 years from when the idea that the square root of a negative number may have meaning, to the definition of i = sqrt(-1).
Historical note	273			Germany, USA		Carl Friedrich Gauss, Charles Proteus Steinmetz		Carries on the narrative from the previous Historical note.
Activity 1	281	Vector geometry						This Activity on the Triangle Inequality links complex numbers with vector geometry.
Activity 2	281-282							This Activity on locus shows how complex number representation can be more efficient than the real Cartesian plane for describing sets of points.
Historical note	290					Leonhard Euler	What is beauty?	Euler's "beautiful" equation links the three great constants of mathematics: exponential e, imaginary i, and the ratio π of a circle's circumference to its diameter.
Investigation	290 (link)	Compound interest						Having previously used the idea of continuously compounding interest to motivate exponential e, we now consider the idea of "imaginary" compound growth. By tracing the movement of a number in the complex plane, we deduce that multiplication by e^iθ produces an anticlockwise rotation in the Argand plane through angle θ. This is compared with multiplication by the real e^r which produces an enlargement.

Chapter 12: Matrices

Investigation 1	311							Students discover the properties of matrix multiplication by observation and then proof of general cases.

	Page	Topic link	Subject link	International link	Cultural link	Historic link	TOK link	Comments
Investigation 2	316-317							Students discover the properties of 2x2 matrix determinants by observation and then proof of general cases.
Activity	322-323							Matrix multiplication provides a simple method of encrypting and deciphering messages.

Chapter 13: Eigenvalues and Eigenvectors

Opening Problem	328	Probability						Rather than a traditional dry analytic opening to a chapter on eigenvalues and eigenvectors, we open here with a Markov Chains problem in context. This reflects the desire of the course to be applications and interpretation oriented. Teachers should be aware that in this course, Markov chains (such as the one presented here) are done unconventionally, as they use a column state matrix and we pre-multiply by a transition matrix which is transpose to convention. We presume the syllabus writers did this in order to connect the long-term behaviour to the eigenvectors of the transition matrix. However, somewhat frustratingly, transition in graph theory is done conventionally, with a row state matrix and post-multiplying by a conventional transition matrix.
Historical note	333		Physics			Leonhard Euler, Joseph-Louis Lagrange, David Hilbert		
Historical note	339	Probability				Andrey Markov		
Activity 1	344 (link)	Probability						A hidden Markov chain is not directly observable. We can observe a sequence of events caused by the chain, but not the chain itself.
Exercise 13D.2, q7	347	Probability						Considers the steady state of a general Markov chain with 2x2 transition matrix. Asks students to give physical interpretation for the steady state proportions of the population.
Activity 2	348-349	Probability	Biology	Scotland / Australia		Patrick Leslie		Leslie matrices are transition matrices for age-group based population dynamics. The models they generate are similar to Markov chains.

Chapter 14: Affine transformations

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Opening Problem	354	Fractals						Sierpiński's triangle is a fractal generated by repeated application of a sequence of affine transformations. It will be studied more thoroughly in Activity 3.
Theory	354							The syllabus talks about transformations: stretches, rotations, reflections, translations, and compositions of these. While it does not use the word, these are collectively called affine transformations. We have also defined linear transformations which include stretches, rotations, and reflections, but not translations. We think this is useful because linear transformations have some special properties (for example, uniqueness) and we want to be able to specifically refer to either linear or affine throughout the chapter.
Investigation 1	356	Coordinate geometry, Matrices						Uses the coordinate geometry formulae for basic rotations to generate the corresponding transformation matrices.
Theory	356	Complex numbers						While this course lacks the trigonometric identities for angle sums, we can use the Eulerian identity to derive the transformation matrix for a rotation through angle θ.
Investigation 2	359	Coordinate geometry, Matrices						Uses the coordinate geometry formulae for basic reflections to generate the corresponding transformation matrices.
Theory	359	Complex numbers						While this course lacks the trigonometric identities for angle sums, we can use the Eulerian identity to derive the transformation matrix for a reflection in the line $y = (\tan\alpha)x$.
Exercise 14F, q1	364	Matrices						Students should recognise that since, in general, $AB \neq BA$, the order in which linear transformations are applied is important.
Investigation 3	365-366	Complex numbers						This investigation focuses on the special properties of linear transformations, in particular the significance of the eigenvalues and eigenvectors of their transition matrices. We reinforce the connection between e^iθ and rotations.
Activity 1	366-367							This introduction to repeated affine transformations highlights what translations add to linear transformations, and may motivate how affine transformations could be used to generate patterns or fractals.
Investigation 4	367-368	Measurement						Students discover the effect of affine transformations on area.
Activity 2	370							Students discover the effect of linear transformations on sense.

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Activity 3	370-372	Fractals			Patterns and designs	Benoit Mandelbrot, John Hutchinson, Michael Barnsley		This Activity brings together the themes of this chapter, and considers three fractals generated by affine transformations: - Sierpiński's triangle introduced in the Opening Problem - von Koch's curve (please note the errata on our website for this section, which will be incorporated into the next print run) - the Barnsley fern which includes a probability component.

Chapter 15: Graph Theory

Activity 1	382 (link)		Computer Science		Internet			Students see how the PageRank algorithm used by the search engine Google uses weighted directed graphs.
Theory of Knowledge	384	Maps		England	Transportation	Henry Beck	Design, Technology	Beck's map for the London Tube has developed into one of the most recognised maps on the planet. Is function and application more important than accuracy? Does technology remove the requirement for knowledge?
Investigation	387							Students discover the adjacency matrix for a multi-step route.
Discussion	392							In this discussion we compare and contrast the transition matrices for graph theory with those for the Markov chains seen earlier. It is imperative that students understand matrix multiplication so they can use transition matrices in either orientation.
Activity 2	401			Prussia		Leonhard Euler		The Bridges of Königsberg is one of the classic problems of graph theory.
Historical note	407					William Hamilton		

Chapter 16: Voronoi Diagrams

Historical note	424			Ukraine		Georgy Feodosevich Voronoy		
Research	427-428		Biology					Voronoi diagrams in nature
Historical note	441-442	Maps, Statistics	Medicine	England		Louis Pasteur, John Snow	Mathematics underpinning other subjects	If Voronoi had been born before Snow, his diagrams could have made Snow's deduction so much easier.

Chapter 17: Introduction to differential calculus

	Page	Topic link	Subject link	International link	Cultural link	Historic link	TOK link	Comments
Investigation 1	452-453	Kinematics						Uses the real-world context of velocity to motivate a study of instantaneous rate of change.
Theory of Knowledge	457		Physics	Ancient Greece		Zeno of Elea	Paradoxes	
Investigation 2	459	Sequences						Considers limits at infinity in the context of a basic sequence. Students should well understand that as the term increases, the sequence gets closer and closer to 1/3.
Historical note	461			Ancient Egypt, Ancient Greece, Europe		Democritus, Eudoxus, Archimedes, Johann Bernoulli, Isaac Barrow		

Chapter 18: Rules of Differentiation

Opening Problem	470	Transformation of functions						The transformation of functions previously studied can give clues to the relationships between derivative functions.
Investigation 1	470-471							Uses first principles with integer powers to deduce the derivative of terms of the form c*x^2. Uses technology to consider the derivative of terms of the form c*x^n where n is an integer. Uses first principles to discover the addition rule for differentiation.
Investigation 2	475-476							Leads to the Chain rule
Investigation 3	477-478							Leads to the Product rule
Investigations 4 & 5	482, 483							Leads to the derivative of e^x
Investigation 6	485							Leads to the derivative of ln x
Investigation 7	488							Leads to the derivatives of sin x and cos x
Exercise 18G q6	490							Calculus derivation of Euler's formula

Chapter 19: Properties of Curves

Exercise 19F q4	518							Properties of the standard normal curve.
Exercise 19F q6,7	518							Properties of the surge and logistic functions first seen in the non-linear modelling section Chapter 8 Activity 2.

Chapter 20: Applications of Differentiation

Theory of Knowledge	537						Optimisation	How does the given problem relate to the law of reflection in Physics?

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Exercise 20C q5	539	Inverse proportion	Physics					This question has the potential to really enlighten students who struggle to connect the mathematics of a model with its physical meaning. If a student is struggling, they could start from the other end: Since the surface area of a sphere is proportional to r^2, we might expect that illuminance might fall off as 1 / r^2.
Activity	539 (link)		Graphic Design, Engineering					Cubic splines are a popular and useful modelling tool.

Chapter 21: Introduction to Integration

Opening Problem	548		Physics			Archimedes		We begin the study of integration by following its historical development.
Investigation 1	550	Series, Limits						Using the same series formula as used in the Core Topics investigation deriving the volume of a tapered solid formula, we prove Archimedes' result for the area under y = x^2 on the interval 0 < x < 1.
Historical note	553					Bonaventura Cavalieri, Sir John Wallis		
Historical note	554					Sir Isaac Newton, Gottfried Wilhelm Leibniz, Bernhard Riemann	Parallel development	The progression from Archimedes to modern calculus was only possible with the introduction of limits.
Exercise 21B q3	555							Links to the standard normal deviation and the proportion of data within 3 standard deviations of the mean.

Chapter 22: Techniques for Integration

Exercise 22A	567							This exercise is built as an Investigation leading to the rules of integration.

Chapter 23: Definite Integrals

Activity 2	604 (link)	Probability				Georges-Louis Leclerc, Comte de Buffon		First historical application of calculus to probability.

Chapter 24: Kinematics

	Page	Topic link	Subject link	International link	Cultural link	Historic link	TOK link	Comments
Discussion	611	Vectors	Physics					From the outset, students can discuss the terminology they have for motion, and how the physics and mathematics relate.
Discussion	628	Chain rule						This discussion is subtle. To help them, consider $a = -9.8$, $v = -9.8t + c$, $s = -4.9t^2 + ct + d$. Using the formula for v, we get $t = (v - c)/(-9.8)$. Substitute this into the formula for s, and hence show that $ds/dv = -9.8v$. Hence $dv/ds = -1/(9.8v)$. So, when $v = 0$, dv/ds is undefined. Applying the chain rule in this case has given us 0 x infinity.
Activity 1	633-634	Modelling, Vectors	Astronomy			Johannes Kepler		Considers the parametric equations for motion in an elliptic orbit. Invites the students to compare our scientific understanding of the orbits of the planets in our solar system with the models of elliptic orbits based on Kepler's laws of planetary motion.
Activity 2	634 (link)		Graphic Design					Bézier curves are constructed as parametric equations which can be compared with the movement between two points.
Investigation	634	Vectors						Students use their knowledge of integration to derive analytic formulae for projectile motion.
Historical note	637		Physics	Italy		Galileo Galilei		

Chapter 25: Differential Equations

Opening Problem	644	Kinematics	Physics			Robert Hooke, Sir Isaac Newton		A mass on a spring is mathematically equivalent to a pendulum. From the outset, our study of differential equations focuses on the physical interpretation of each term of the equation. This is followed up in the theory presented on page 645.
Exercise 25A q5	646	Scientific notation	Physics			Sir Isaac Newton		Links Newton's law of universal gravitation to the commonly accepted acceleration due to gravity for an object much smaller than the Earth at low altitude.
Exercise 25B q8	648	Complex numbers	Physics					Continues the theme of simple harmonic motion. - Verifies trigonometric solutions. - Verifies Euler form complex solution. - Verifies that any linear combination of solutions is also a solution.

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Activity 1	664-665		Biology			Pierre François Verhulst		In the absence of partial fractions, students are not expected to derive the solutions of the logistic growth differential equation. However, having presented this solution, students can apply the analytic model. They can also compare with numerical solutions using Euler's method.
Investigation	665 (link)	Numerical methods				Leonhard Euler		In this Investigation, students consider an improvement of Euler's method. They should gain an understanding that the step size is always important, but so is the accuracy of the model. Not everything can be fixed by reducing the step size.
Activity 2	665 (link)		Biology					In this Activity students consider the population dynamics of the spruce budworm. They study equilibrium points with the aid of a phase line. This is the one-dimensional equivalent of the two-dimensional phase portraits they will use for coupled systems in Chapter 26. This could make this Activity particularly useful.
Theory of Knowledge	665-666	Chaos theory				Sir Isaac Newton, Leonhard Euler, Pierre-Simon Laplace, Jean-Baptiste Fourier, Edward Lorenz	Technology, Analytic vs Numerical methods	Are we inevitably limited in our understanding of the world?

Chapter 26: Coupled Differential Equations

Opening Problem	670		Biology			Alfred Lotka, Andrey Kolmogorov, Vito Volterra		Introduces the Lotka-Volterra predator-prey model, which will be a theme picked up in Activity 1.
Activity 1	675-677		Biology					In this Activity students consider several population models. In each case there is emphasis on understanding the physical meaning of each term in the coupled differential equations. - The Lotka-Volterra predator-prey model - a more sophisticated Kolmogorov predator-prey model which includes extra terms that cause the solution to spiral in to the central equilibrium point - a competing species model with 4 equilibrium points.

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Investigation 1	678-679	Eigenvalues and Eigenvectors						For coupled linear DEs, students discover the relationship between the eigenvectors of the matrix of coefficients and trajectories which are straight lines. They will also see how complex eigenvalues result in rotation in the phase portrait.
Investigation 2	680-681							Students discover how the eigenvalues of the transition matrix result in the different possible forms of equilibrium point, including straight line trajectories, rotation, and stability.
Activity 2	686 (link)		Chemistry					Coupled differential equations are observed in all the sciences. Again, there is a focus on physical understanding and interpretation of solutions.
Example 4	687		Physics, Electrical engineering			Balthasar van der Pol		
Activity 3	688 (link)		Physics					Students analyse the phase portrait for a simple rigid pendulum. They should note carefully that rather than a horizontal displacement x which is a reasonable approximation for small angles of displacement, we consider here the angular displacement directly. This allows for a solution which is pushed past horizontal by sufficient initial velocity.
Exercise 26D, q6	692					Pierre Bouguer		With the aid of numerical methods, students can approximate solutions to quite complicated systems. In this question they consider a radiodrome or curve of pursuit.
Activity 4	692 (link)		Biology		Vaccines			The SIR model for the spread of infectious disease can model the epidemiology of some infections with extreme accuracy. It shows how a mathematical model can be built up using extra terms in the differential equations. Students will learn to model several types of disease outbreak, and understand the efficacy of treatment and vaccination. They will also learn how a disease may become endemic in a population, and why future outbreaks occur. This exciting Activity could certainly be extended to a Mathematical Exploration or Extended Essay.

Chapter 27: Discrete Random Variables

Activity	708				Game strategy			
Investigation 2 Investigation 3	718 721							Use of technology to investigate the binomial distribution.

	Page	Topic link	Subject link	International link	Cultural link	Historic link	TOK link	Comments

Chapter 28: The Normal Distribution

Investigation 1	735	Calculus						Investigates the normal curve using differential calculus.
Historical note	736					Carl Friedrich Gauss		
Investigation 3	743-744							This Investigation gives a practical opportunity for students to understand z-scores and the Z-distribution.
Investigation 4	750-751							The normal approximation to the binomial distribution.

Chapter 29: Estimation and Confidence Intervals

Investigation 1	757							Students discover the expectation and variance of a linear combination of random variables.
Investigation 2	762-763							Students explore the distribution of sample means, and in the process discover the Central Limit Theorem.
Theory of Knowledge	767-768			Europe		Abraham de Moivre, Pierre-Simon Laplace, Pafnuty Chebyshev, Andrey Markov, Aleksandr Lyapunov, Sir Francis Galton, George Pólya	Beauty and order, Rationalism and Empiricism	
Activity	771							Students gain a practical understanding of confidence intervals.
Investigation 4	776							Students explore the T-distribution. They should see that it has a bell-shape like the standard normal distribution, but that its proportions of values in the intervals [-1, 1], [-2, 2], [-3, 3] are different.

Chapter 30: Hypothesis Testing

Discussion	787							While appearing simple, it is vital that students fully understand that being confident to a particular degree does not mean that with absolute certainty, the claim is true.
Investigation 1	791-792							Continuing with the theme from the previous discussion, students should understand the importance of multiple testing, because there is probability of a type 1 error α with each test.

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Theory of Knowledge	792		Medicine	England		Sir Ronald Aylmer Fisher	Ethics	
Historical note	797		Agriculture	England, Ireland		William Sealy Gosset, Karl Pearson, Sir Ronald Aylmer Fisher		
Summary	824 (link)							The flowchart is extremely valuable for helping students understand: - the common structure of the statistical hypothesis tests - which test to use - which variables each test uses.

Chapter 31: $\chi 2$ **Hypothesis Tests**

Activity	837		Genetics			Gregor Johann Mendel		
Theory of Knowledge	838		Genetics, Medicine		Vaccines	Sir Ronald Aylmer Fisher	Ethics	Fisher concluded that the data presented by Mendel, “the father of modern genetics” was too perfect to have been true. The MMR vaccine hoax has been cited as “perhaps the most damaging medical hoax in the last 100 years”, having given rise to the anti-vaccine movement.